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Review

Projective connections

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Abstract

We compare the approaches of E. Cartan and of T.Y. Thomas and J.H.C. Whitehead to the study of 'projective connections'. Although the quoted phrase has quite different meanings in the two contexts considered, we show that a class of projectively equivalent symmetric affine connections – or, more generally, sprays – on a manifold (the latter meaning) gives rise, in a global way, to a unique Cartan connection on a principal bundle over the manifold (the former meaning). The principal bundle on which the Cartan connection is defined is itself a geometric object, and exists independently of any particular connection. In the course of the discussion we derive a Cartan normal projective connection for a system of second-order ordinary differential equations (extending the results of Cartan from a single equation to many) and we generalize the concept of a normal Thomas–Whitehead connection from affine to general sprays.

MSC: 53B10

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Contents

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1. Introduction

In this paper we compare the approaches of E. Cartan on the one hand, and T.Y. Thomas and J.H.C. Whitehead on the other, to the study of projective connections. There has recently been a resurgence of interest in both of these approaches, with a view to applications and for purely mathematical reasons. Cartan's approach to connection theory and the equivalence of geometric structures has been found to be relevant to the programme of research in general relativity which has been carried out over the last dozen years by Newman and his co-workers (see [12]). As a consequence Cartan's theory of projective connections has been subject to new scrutiny (see [19] and [20]). There are also applications in the theory of projectively equivariant quantizations: see [14] and references therein. We should also mention recent work on the geometrical study of differential equations using Cartan's methods in [9,11,13]. The approach of Thomas and Whitehead, on the other hand, has been discussed in [1], also from a relativistic perspective. So far as purely mathematical interest in Cartan is concerned there is the book of Sharpe [23], and a considerable body of work on so-called parabolic geometries, of which we take [2] as representative. A modern version of the method used by Thomas and Whitehead, captured in the concept of a Thomas–Whitehead projective connection, has been given by Roberts in [21] and developed in [15].

Elie Cartan's paper on projective connections [3], published in 1924, was one of a series intended to extend the idea of an affine connection as formulated by Levi-Civita and Weyl to a more general, non-vector, situation. Cartan imagined, attached to each point of a manifold, a projective space of the same dimension, together with a mechanism whereby the spaces at two neighbouring points could be 'connected'. Such a connection would define geodesics as those curves in the manifold which could be 'developed' into straight lines in the connected projective spaces.

A modern interpretation of Cartan's idea can be found in the recent book by Sharpe [23]. According to Sharpe, the fruitful way to view Cartan's theory of connections is to think of it as a generalization of Klein's concept of geometry. In this approach, each 'Cartan geometry' is based upon a model geometry called a 'Klein geometry'. A Klein geometry is a homogeneous space G/H of a Lie group G; G itself is a principal H-bundle over G/H and comes equipped with a $\mathfrak{g}$-valued 1-form (where $\mathfrak{g}$ is the Lie algebra of G), its Maurer–Cartan form. A Cartan geometry on a manifold M, corresponding to a Klein geometry for which G/H has the same dimension, is a principal H-bundle $P \to M$ together with a $\mathfrak{g}$-valued 1-form ω on P with certain properties, which is called the 'connection form' and is intended to generalize the Maurer–Cartan form. A construction of this kind is called a Cartan connection.

The concept of a Cartan connection can be reformulated in terms of objects defined locally on the base manifold M. First, suppose given a principal bundle P and $\mathfrak{g}$-valued 1-form ω as described above; take a covering of M by open sets $\{U\}$ over each of which P is trivial, and for each U a local section κ of $P|_U \to U$; then $\omega_U = \kappa^*\omega$ is a

$\mathfrak{g}$-valued 1-form on U, and for any two sets U, V of the cover we have $\omega_V = \mathrm{ad}(h^{-1})\omega_U + h^*\theta$ on $U \cap V$, where $h : U \cap V \to H$ is the transition function for P and θ is the Maurer–Cartan form of H. The local section κ is called a gauge, the local 1-form ω_U the gauged connection form, and the relation between gauged connection forms the gauge transformation rule. Conversely, given an open covering of M, for each member U of the covering a $\mathfrak{g}$-valued 1-form ω_U on U, and for each pair U, V of members of the covering with non-empty intersection a map $h : U \cap V \to H$ such that $\omega_V = \mathrm{ad}(h^{-1})\omega_U + h^*\theta$ on $U \cap V$, one can construct a principal H-bundle $P \to M$ of which the functions h are the transition functions, and a $\mathfrak{g}$-valued 1-form ω on P of which the ω_U are gauged representatives. It will be important for us in our discussion of Cartan connections that one can pass at will between the global description and the local, gauged, description.

Cartan connections differ in concept and in practice from the type of connection on a principal bundle introduced in 1950 by Ehresmann. Ehresmann's definition of a connection is based on the idea of parallel transport originally formulated by Levi-Civita; on the face of it, there is no notion of parallelism associated with a Cartan projective connection. The practical differences show up in the fact that the connection form of a Cartan connection takes its values in $\mathfrak{g}$ while that of an Ehresmann connection takes its values in the Lie algebra of H, the group of the principal bundle.

The simpler of the two examples of Cartan connections that we will discuss is the Cartan projective connection described in the first part of [3]. For a Cartan projective connection on an m-dimensional manifold the model geometry is m-dimensional real projective space P^m. To realise this as a homogeneous space G/H we take for G the group of projective transformations of P^m, which is $\mathrm{PGL}(m+1)$, the quotient of $\mathrm{GL}(m+1)$ by non-zero multiples of the identity; and for H we take the subgroup $\mathrm{H}_{m+1} \subset \mathrm{PGL}(m+1)$ which is the stabilizer of the point $[1, 0, \ldots, 0] \in \mathrm{P}^m$.

Cartan's work on connections predates the formulation of the concept of a fibre bundle, of course, so though he discusses in detail the projective connection as a local object, there is no direct hint in [3] of what the principal H_{m+1}-bundle on which the global connection form should live might be (except of course that it should embody the notion of attaching a projective space to each point of the manifold). In [23] Sharpe gives the general procedure for constructing the bundle implicitly by inferring its transition functions from the local connection forms, and shows that it is unique to within equivalence, but he does not carry out the specific construction for the particular case of the projective connection, nor give an explicit definition of the bundle. One aim of this paper is to make good this deficiency.

Around the time that Cartan published his paper on projective connections a somewhat different line of research, also described as a theory of projective connections, was being pursued by several other authors, including Thomas [25,26] and Whitehead [27]. This second theory is concerned with the relationship between two affine connections whose geodesics, although having different parametrizations, are geometrically the same; two such connections are said to be projectively related. Here the concept of connection is that of Ehresmann. A brief history of the development of these ideas up to 1930, which names the mathematicians principally involved, can be found in the introductory section of Whitehead's paper.

A generalization of the projective differential geometry of affine connections was studied by Douglas [10] under the name of the general geometry of paths; it is also known as the projective differential geometry of sprays. We will extend the Thomas–Whitehead theory to cover this more general situation. The projective differential geometry of sprays incorporates the geometrical theory of systems of second-order ordinary differential equations under so-called point transformations. We are therefore able to interpret the point transformation invariants of a system of second-order ordinary differential equations found by Fels [11] in terms of projective quantities associated with the corresponding equivalence class of sprays.

In the following sections of this paper we review spray geometry, discuss the Thomas–Whitehead construction and its generalization to sprays, and afterwards describe the Cartan theory in the affine case. We then introduce a certain bundle which realizes explicitly Cartan's idea of attaching a projective space to each point of a manifold, and which we call the Cartan bundle; this bundle is defined independently of any particular choice of connection, but to any projective class of sprays one can associate a unique Cartan connection on the Cartan bundle, as we show. Finally we describe in detail how to derive the Cartan connection from the generalized Thomas–Whitehead data for any projective equivalence class of sprays.

We will use the Einstein summation convention, over several different ranges of indices. We will be basically concerned with a manifold of dimension m, and we will denote by $a, b, \ldots$, indices which range and sum over $1, 2, \ldots, m$. We will also need to use indices which range and sum over $0, 1, \ldots, m$: we will denote such indices by

$\alpha, \beta, \ldots$. Finally, we will sometimes single out the index 1 for special attention, and when we do so indices which range and sum over $2, 3, \ldots, m$ will be denoted by $i, j, \ldots$.

2. Projective differential geometry of sprays

We review here the projective geometry of sprays. A useful reference for this material is Shen's book [24]; however, our approach differs from his in that we put more emphasis on the similarities between the general case and the affine case as described for example in Schouten's 'Ricci-Calculus' [22]. Douglas [10] also covers much of this ground of course.

2.1. Sprays and Berwald connections

We denote by $\tau_M^\circ : T^\circ M \to M$ the slit tangent bundle of M (TM with the zero section deleted). Coordinates on $T^\circ M$ will generally be written (x^a, u^a). The Liouville field $u^a\partial/\partial u^a$ is denoted by Δ.

A spray S on M is a second-order differential equation field on $T^\circ M$,

$$S = u^a \frac{\partial}{\partial x^a} - 2\Gamma^a \frac{\partial}{\partial u^a},$$

whose coefficients Γ^a are positively homogeneous of degree 2 in the u^a; if they are quadratic in the u^a (so that S is the geodesic field of a symmetric affine connection) then the spray is said to be affine.

Homogeneity occurs frequently and is always with respect to the u^a, so we will just say, for example, that some function is of degree 1. Moreover, the distinction between being positively homogeneous and being homogeneous without qualification won't be important in this subsection, so we won't repeat the qualifier 'positively'.

The horizontal distribution associated with a spray is spanned by the vector fields

$$H_a = \frac{\partial}{\partial x^a} - \Gamma_a^b \frac{\partial}{\partial u^b}, \quad \Gamma_a^b = \frac{\partial \Gamma^b}{\partial u^a};$$

Γ_a^b is of degree 1. The spray itself is horizontal: $S = u^a H_a$. It will often be convenient to denote the vertical vector field $\partial/\partial u^a$ by V_a.

The Berwald connection (see for example [4]) associated with a spray S is a connection on the pullback bundle $\tau_M^{\circ *}(TM) \to T^\circ M$. We will use tensor calculus methods, so we write sections of $\tau_M^{\circ *}(TM)$ as $X^a\partial/\partial x^a$ where the coefficients X^a are local functions on $T^\circ M$. The Berwald connection can be specified by giving its covariant differentiation operator ∇ operating on $\partial/\partial x^a$ (regarded as a local section of $\tau_M^{\circ *}(TM)$, or vector field along the projection τ_M°), together with the usual rules of covariant differentiation: in fact

$$\nabla_{H_a} \frac{\partial}{\partial x^b} = \Gamma_{ab}^c \frac{\partial}{\partial x^c}, \quad \nabla_{V_a} \frac{\partial}{\partial x^b} = 0,$$

where the connection coefficients are given by

$$\Gamma_{ab}^c = \frac{\partial \Gamma_a^c}{\partial u^b} = \frac{\partial^2 \Gamma^c}{\partial u^a \partial u^b};$$

they are symmetric, of degree 0, and reduce to the usual connection coefficients in the affine case.

Note that covariant differentiation with respect to the vertical vector field V_a of any tensor field along τ_M° amounts simply to partial differentiation of the components of the field with respect to u^a; and that therefore if one takes a tensor field along τ_M° and partially differentiates its components with respect to the u^a one obtains another tensor field, with one more covariant index.

We will use index notation, so that (for example) if T is a type (1, 1) tensor along τ_M° and ξ a vector field on $T^\circ M$, $(\nabla_\xi T)_b^a$ is just written $\nabla_\xi T_b^a$. Moreover, where convenient we will denote by (for example) $T_{b,c}^a$ the tensor component $\nabla_{V_c} T_b^a$, and $T_{b|c}^a$ the tensor component $\nabla_{H_c} T_b^a$.

The so-called total derivative $\mathbf{T}$ is the vector field along τ_M° whose coordinate representation is $u^a\partial/\partial x^a$; its covariant derivative in any horizontal direction vanishes, which is to say that $u_{|b}^a = 0$.

The curvature of the connection is defined in the usual way, but can be broken down into various components according to whether the vector field arguments are taken to be horizontal or vertical. First, evidently

$$\left(\nabla_{V_a}\nabla_{V_b} - \nabla_{V_b}\nabla_{V_a} - \nabla_{[V_a,V_b]}\right)\frac{\partial}{\partial x^c} = 0.$$

Next, we have

$$\left(\nabla_{V_a}\nabla_{H_b} - \nabla_{H_b}\nabla_{V_a} - \nabla_{[V_a,H_b]}\right)\frac{\partial}{\partial x^c} = B^d_{cab}\frac{\partial}{\partial x^d},$$

where (since $[V_a, H_b]$ is vertical)

$$B^d_{cab} = \frac{\partial \Gamma^d_{bc}}{\partial u^a} = \frac{\partial^3 \Gamma^d}{\partial u^a \partial u^b \partial u^c}.$$

This component of the curvature has no affine counterpart—in fact its vanishing is the necessary and sufficient condition for the spray to be affine. It is completely symmetric in the lower indices, is homogeneous of degree -1, and satisfies $B^d_{cab}u^c = 0$. It is called the Berwald curvature.

Finally,

$$\left(\nabla_{H_a}\nabla_{H_b} - \nabla_{H_b}\nabla_{H_a} - \nabla_{[H_a,H_b]}\right)\frac{\partial}{\partial x^c} = R^d_{cab}\frac{\partial}{\partial x^d},$$

where R^d_{cab}, the counterpart of the usual curvature, is given by

$$R^d_{cab} = H_a\left(\Gamma^d_{bc}\right) - H_b\left(\Gamma^d_{ac}\right) + \Gamma^d_{ae}\Gamma^e_{bc} - \Gamma^d_{be}\Gamma^e_{ac}.$$

It has the usual symmetries, is of degree 0, and reduces to the ordinary curvature tensor when the spray is affine. It is called the Riemann curvature.

We can also express the curvatures conveniently using forms. We write φ^a for the 1-form $\mathrm{d}u^a + \Gamma^a_b \mathrm{d}x^b$, so that $\{\mathrm{d}x^a, \varphi^a\}$ is the local basis of 1-forms on $T^\circ M$ dual to the local basis $\{H_a, V_a\}$ of vector fields. Define connection forms $\omega^a_b = \Gamma^a_{bc}\mathrm{d}x^c$; if $\Omega^a_b = \mathrm{d}\omega^a_b + \omega^a_c \wedge \omega^c_b$ are the associated curvature forms then

$$\Omega^a_b = \frac{1}{2}R^a_{bcd}\mathrm{d}x^c \wedge \mathrm{d}x^d + B^a_{bcd}\varphi^c \wedge \mathrm{d}x^d.$$

By taking traces of the curvatures we obtain tensors $B_{ab} = B^c_{cab}$ and $R_{ab} = R^c_{acb}$. The first is symmetric. The second is not in general symmetric; moreover, by the cyclic identity $R^c_{cab} = R_{ab} - R_{ba}$.

By differentiating the formula for the Riemann curvature with respect to u^e one obtains the following relation between the two curvatures:

$$R^d_{cab,e} = B^d_{bce|a} - B^d_{ace|b};$$

this is in fact part of the second Bianchi identity for the curvature taken as a whole. From this formula, by taking a trace and skew-symmetrizing over one pair of indices one obtains $R_{ab,c} - R_{ba,c} = B_{bc|a} - B_{ac|b}$, and by skew-symmetrizing over a different pair of indices $R_{bc,a} = R_{ac,b}$, both of which are useful later.

We will also be concerned with the associated tensor

$$R^a_b = R^a_{cbd}u^c u^d = 2\frac{\partial \Gamma^a}{\partial x^b} - S(\Gamma^a_b) - \Gamma^a_c\Gamma^c_b.$$

This type (1, 1) tensor field is often called the Jacobi endomorphism, because it is the curvature term that appears in the Jacobi equation. It contains the same information as the Riemann tensor, which can be recovered from it by use of the formula

$$R^d_{cab} = \frac{1}{3}\left(R^d_{a,bc} - R^d_{b,ac}\right).$$

We denote by R the trace of R^a_b; we have $R = R_{cd}u^c u^d$. It follows from the relationship $R_{bc,a} = R_{ac,b}$ and the fact that R_{ab} is homogeneous of degree 0 that $u^b R_{bc,a} = 0$, whence

$$\frac{\partial R}{\partial u^a} = (R_{ab} + R_{ba})u^b, \quad \frac{\partial^2 R}{\partial u^a \partial u^b} = R_{ab} + R_{ba}.$$

A spray whose Jacobi endomorphism has the property that for any v^a, $R^a_b v^b$ is a linear combination of u^a and v^a is said to be isotropic. For an isotropic spray R^a_b takes the form $R^a_b = \lambda\delta^a_b + \mu_b u^a$ for some scalar λ and vector μ_b. Since $R^a_b u^b = 0$ we have $\lambda = -\mu_b u^b$, and then by taking the trace we find that $(m-1)\lambda = R$, so for an isotropic spray

$$R^a_b - \frac{1}{m-1} R\delta^a_b = \mu_b u^a,$$

with $\mu_b u^b = -R/(m-1)$.

Note that a spray has two independent curvatures, and either can vanish without the other doing so. A spray whose Riemann curvature vanishes, but whose Berwald curvature is not necessarily zero, is said to be R-flat.

2.2. Projective equivalence

Two sprays S, $\hat{S}$ are projectively equivalent if $\hat{S} - S = -2\alpha\Delta$, or $\hat{\Gamma}^a = \Gamma^a + \alpha u^a$, where the function α is positively homogeneous of degree 1 in the u^a.

From the basic projective transformation rule it follows that the horizontal vector fields associated with the spray $\hat{S}$ are given by

$$\hat{H}_a = H_a - \alpha V_a - \alpha_a \Delta, \qquad \alpha_a = \frac{\partial \alpha}{\partial u^a};$$

α_a is of degree 0, and $u^a\alpha_a = \alpha$. Furthermore,

$$\hat{\Gamma}^c_{ab} = \Gamma^c_{ab} + \left(\alpha_{ab}u^c + \alpha_a\delta^c_b + \alpha_b\delta^c_a\right), \qquad \alpha_{ab} = \frac{\partial^2\alpha}{\partial u^a \partial u^b};$$

α_{ab} is symmetric and of degree -1, and $\alpha_{ab}u^b = 0$.

By taking a trace in the equation for the transformation of the Γ^c_{ab}, and writing Γ_a for Γ^b_{ab}, we obtain $\hat{\Gamma}_a = \Gamma_a + (m+1)\alpha_a$, whence the quantity

$$\Pi^c_{ab} = \Gamma^c_{ab} - \frac{1}{m+1}\left(\Gamma_a\delta^c_b + \Gamma_b\delta^c_a + B_{ab}u^c\right)$$

is projectively invariant. Douglas [10] calls it the fundamental invariant and says in effect that every projective invariant is expressible in terms of it and its partial derivatives. However, the Π^c_{ab} are not components of a tensor, nor even of a connection, and this has to be borne in mind when forming projective invariants from it. Note that $\Pi^b_{ab} = \Pi^b_{ba} = 0$.

It may appear that if we set $\Gamma = \Gamma^d_d$ and take

$$\alpha = -\frac{1}{m+1}\Gamma = -\frac{1}{m+1}\frac{\partial \Gamma^d}{\partial u^d}$$

then the transformed spray has Π^c_{ab} for its connection coefficients. However, Γ is not strictly speaking a function: its transformation law under coordinate transformations of the x^a (and the induced transformations of the u^a) involves the determinant of the Jacobian of the coordinate transformation; however, it transforms as a function under coordinate transformations for which the determinant of the Jacobian is 1. Consequently the Π^c_{ab} are not, in general, the components of a connection. In fact if Π^c_{ab}, $\hat{\Pi}^c_{ab}$ are the components of the fundamental descriptive invariant with respect to coordinates (x^a), $(\hat{x}^a)$ then

$$\hat{\Pi}^c_{ab} = \bar{J}^d_a \bar{J}^e_b (J^c_f \Pi^f_{de} - J^c_{de}) + \frac{1}{m+1}\frac{\partial \log|J|}{\partial x^d}(\bar{J}^d_a\delta^c_b + \bar{J}^d_b\delta^c_a),$$

where $J^a_b = \partial\hat{x}^a/\partial x^b$ are the elements of the Jacobian matrix of the coordinate transformation, $\bar{J}^a_b$ those of its inverse, $J^c_{ab} = \partial J^c_a/\partial x^b = \partial J^c_b/\partial x^a$, and J is the Jacobian determinant.

We will nevertheless follow Douglas in taking the Π^c_{ab} as fundamental in describing a certain kind of path space, that is, a manifold together with a collection of paths (unparametrized curves) with the property that there is exactly one path of the collection through a given point in a given direction.

We concentrate for a moment on the affine case, when the Π^c_{ab} are functions on a coordinate patch in the base manifold M. Douglas, in [10], calls a path space of this kind a restricted path space. In fact we could define a restricted path space as an assignment, to each coordinate patch on a manifold, of a set of functions Π^c_{ab}, symmetric in a and b, transforming under a change of coordinates according to the formula given above. The paths are defined by

$$\ddot{x}^c + \Pi^c_{ab}\dot{x}^a\dot{x}^b \propto \dot{x}^c,$$

a condition which is invariant both under coordinate transformations and under change of parametrization. It is not strictly necessary to impose the condition that $\Pi_a = \Pi^b_{ab} = 0$, since if the Π^c_{ab} transform as specified then the Π^b_{ab} are components of a 1-form, and if

$$\tilde{\Pi}^c_{ab} = \Pi^c_{ab} - \frac{1}{m+1}\left(\Pi_a\delta^c_b + \Pi_b\delta^c_a\right)$$

then $\tilde{\Pi}^c_{ab}$ transforms in the same way, defines the same paths, and does satisfy $\tilde{\Pi}_a = 0$. Nevertheless we will reserve the term 'fundamental descriptive invariant' for the Π^c_{ab} which satisfy $\Pi_a = 0$. Clearly if $\Pi_a = 0$ and $\hat{\Pi}^c_{ab}$ is related to Π^c_{ab} by the transformation formula given above then $\hat{\Pi}_a = 0$ also.

Every affine connection defines a restricted path space in this sense, with projectively equivalent ones defining the same path space. As it happens the converse also holds, as we will show later, so the concept of restricted path space is not more general.

2.3. Projective transformation of the curvatures

We now return to the general case, and derive certain projective transformation formulae we require. We will do so entirely tensorially; we will however point out the simplifications that arise when one chooses the local spray whose connection coefficients with respect to some coordinates are the Π^c_{ab}. We will denote objects calculated in this way by setting their kernel letters in black-letter. Thus it follows from the observation above that the traces of Π^c_{ab} vanish that $\mathfrak{B}_{ab} = 0$, and also that $\mathfrak{R}^c_{cab} = 0$, so that $\mathfrak{R}_{ba} = \mathfrak{R}_{ab}$.

An easy calculation leads to the following transformation formula for B^d_{cab}:

$$\hat{B}^d_{cab} = B^d_{cab} + \alpha_{abc}u^d + \alpha_{ab}\delta^d_c + \alpha_{bc}\delta^d_a + \alpha_{ac}\delta^d_b,$$

where α_{abc} denotes a third partial derivative of α; it satisfies $\alpha_{abc}u^c = -\alpha_{ab}$. Then by taking a trace $\hat{B}_{ab} = B_{ab} + (m+1)\alpha_{ab}$, whence

$$D^d_{cab} = B^d_{cab} - \frac{1}{m+1}\left(u^d B_{ab,c} + B_{ab}\delta^d_c + B_{bc}\delta^d_a + B_{ac}\delta^d_b\right)$$

is a projectively invariant tensor—the Douglas tensor. It is symmetric in its lower indices, of degree 0, it satisfies $D^d_{cab}u^c = 0$, and all of its traces vanish. Since $\mathfrak{B}_{ab} = 0$,

$$D^d_{cab} = \mathfrak{B}^d_{cab} = \frac{\partial \Pi^d_{ab}}{\partial u^c}.$$

The projective transformation of the Riemann curvature is given by

$$\hat{R}^d_{cab} = R^d_{cab} + \nabla_{H_a}\alpha^d_{bc} - \nabla_{H_b}\alpha^d_{ac} + (\alpha\alpha_{bc} + \alpha_b\alpha_c)\delta^d_a - (\alpha\alpha_{ac} + \alpha_a\alpha_c)\delta^d_b,$$

where α^c_{ab} is the difference tensor of the connection coefficients,

$$\alpha^c_{ab} = \alpha_{ab}u^c + \alpha_a\delta^c_b + \alpha_b\delta^c_a.$$

Using the fact that multiplying by u^a commutes with covariant differentiation with respect to H_a we obtain

$$\nabla_{H_a}\alpha^d_{bc} = u^d\alpha_{bc|a} + \alpha_{b|a}\delta^d_c + \alpha_{c|a}\delta^d_b.$$

It can be shown that

$$\nabla_{H_a}\hat{B}_{bc} - \nabla_{H_b}\hat{B}_{ac} = \hat{\nabla}_{\hat{H}_a}\hat{B}_{bc} - \hat{\nabla}_{\hat{H}_b}\hat{B}_{ac},$$

whence the transformation law for R^d_{cab} can be rewritten in the form

$$\hat{S}^d_{cab} = S^d_{cab} - A_{bc}\delta^d_a + A_{ac}\delta^d_b + (A_{ab} - A_{ba})\delta^d_c,$$

where

$$\begin{aligned} S^d_{cab} &= R^d_{cab} - \frac{1}{m+1}u^d\left(B_{bc|a} - B_{ac|b}\right) \\ &= R^d_{cab} - \frac{1}{m+1}u^d\left(R_{ab,c} - R_{ba,c}\right), \end{aligned}$$

$$A_{ab} = \alpha_{b|a} - \alpha\alpha_{ab} - \alpha_a\alpha_b.$$

The modified Riemann curvature S^d_{cab} has the usual symmetries, is of degree 0, and reduces to the Riemann curvature in the affine case. We set $S_{ab} = S^c_{acb}$; then $\hat{S}_{ab} = S_{ab} + A_{ab} - mA_{ba}$, whence

$$A_{ab} = -\frac{1}{m^2-1}\left(\hat{Q}_{ab} - Q_{ab}\right), \quad Q_{ab} = S_{ab} + mS_{ba}.$$

It follows that

$$P^d_{cab} = S^d_{cab} - \frac{1}{m^2-1}\left(Q_{bc}\delta^d_a - Q_{ac}\delta^d_b - (Q_{ab} - Q_{ba})\delta^d_c\right)$$

is a projectively invariant tensor. It is the counterpart of the projective curvature tensor of the affine theory, to which it reduces in the affine case. It is of degree 0; it has the same symmetries as the Riemann curvature, and in addition all of its traces vanish.

Since $\mathfrak{R}_{ab}$ is symmetric, $\mathfrak{S}^d_{cab} = \mathfrak{R}^d_{cab}$, whence $\mathfrak{S}_{ab} = \mathfrak{R}_{ab}$ and $\mathfrak{Q}_{ab} = (m+1)\mathfrak{R}_{ab}$, so that

$$P^d_{cab} = \mathfrak{R}^d_{cab} - \frac{1}{m-1}\left(\mathfrak{R}_{bc}\delta^d_a - \mathfrak{R}_{ac}\delta^d_b\right).$$

The Jacobi endomorphism of a spray S transforms as follows:

$$\hat{R}^a_b = R^a_b + A_b u^a - A\delta^a_b,$$

where the vector A_a and scalar A are given by

$$A_a = 2H_a(\alpha) - \nabla_S\alpha_a - \alpha\alpha_a, \quad A = S(\alpha) - \alpha^2 = u^a A_a;$$

A_a is homogeneous of degree 1. For the trace of the Jacobi endomorphism we have $\hat{R} = R - (m-1)A$. Using these formulae one can show that

$$W^a_b = R^a_b - \frac{1}{m-1}R\delta^a_b - \frac{1}{m+1}u^a\nabla_{V_c}\left(R^c_b - \frac{1}{m-1}R\delta^c_b\right)$$

is projectively invariant. It is called the Weyl tensor. It is trace-free and satisfies $W^a_b u^b = 0$. The Weyl tensor bears the same relationship to the projective curvature tensor as the Jacobi endomorphism does to the Riemann curvature: that is to say, $P^a_{cbd}u^c u^d = W^a_b$, and P^a_{bcd} can be expressed in terms of second vertical covariant derivatives of W^a_b.

From the transformation laws it is easy to see that being isotropic is a projectively invariant property of a spray. Moreover, by substituting the formula for R^a_b for an isotropic spray into the expression for W^a_b above and using the evident fact that μ_b is homogeneous of degree -1 we see that $W^a_b = 0$ for an isotropic spray; the converse is obvious. Thus a spray is isotropic if and only if $W^a_b = 0$; and equivalently if and only if $P^a_{bcd} = 0$.

When $m > 2$ the vanishing of both the Douglas and the projective curvature tensors is the necessary and sufficient condition for a spray to be projectively flat, that is, projectively equivalent to a spray that can be written $u^a\partial/\partial x^a$ in some coordinates. In dimension 2, however, a tensor with the symmetries of the Riemann tensor is determined by its traces, and if they vanish so does the tensor; so the projective curvature tensor is identically zero in dimension 2. We will therefore assume that $m > 2$ hereafter.

The vanishing of the Douglas tensor alone is the necessary and sufficient condition for a spray to be projectively equivalent to an affine one [24]. The vanishing of the projective curvature tensor alone is the necessary and sufficient condition for a spray to be projectively equivalent to one which is R-flat; in other words, a spray is isotropic if and only if it is projectively R-flat [5].

2.4. Systems of differential equations

The base integral curves of a spray are the solutions of the equations

$$\ddot{x}^a + 2\Gamma^a(x, \dot{x}) = 0;$$

all sprays in a projective equivalence class have the same base integral curves up to a change of parameter which preserves sense. Thus a projective equivalence class of sprays determines, and in fact is determined by, a path space, that is, a collection of paths (unparametrized but oriented curves) in M with the property that there is a unique path of the collection through each point in each direction. A choice of spray in a projective equivalence class amounts to a choice of parametrization of the corresponding paths; Douglas calls the parametrization resulting from the choice with $\Gamma^a_{bc} = \Pi^a_{bc}$ the canonical parametrization for the given coordinates.

Since sprays are required to be only positively homogeneous, reversing the initial direction may give a different path. We will be interested in a restricted class of sprays, those having the property that the integral curve through x with initial tangent vector $-u$ is just the integral curve through x with initial tangent vector u traversed in the opposite sense; we call such sprays, and their base integral curves, reversible. Reversible sprays are such that the coefficients Γ^a are homogeneous of degree 2 without qualification, that is, satisfy $\Gamma^a(x^b, \lambda u^b) = \lambda^2 \Gamma^a(x^b, u^b)$ for all non-zero λ. Alternatively, they satisfy $\Gamma^a(x^b, -u^b) = \Gamma^a(x^b, u^b)$ in addition to being positively homogeneous. The corresponding path space has the property that given a point $x \in M$ and a line in T_xM there is a unique path (now an unparametrized and unoriented curve) through x whose tangent line at x is the given line. The set of lines in T_xM is just $\mathrm{P}T_xM$, the projective tangent space at x; thus a path space in this sense determines and is determined by a congruence of paths on $\mathrm{P}TM$, the projective tangent bundle of M (one and only one path of the congruence passes through each point of $\mathrm{P}TM$); the corresponding projective equivalence class of sprays determines and is determined by a line element field on $\mathrm{P}TM$, the tangent line element field of the congruence of paths.

From here on we will deal only with reversible sprays.

In a local coordinate system we can choose to parametrize suitable paths of a projective class of sprays with one of the coordinates, say x^1; with such a parametrization $\dot{x}^1 = 1$, $\ddot{x}^1 = 0$, and the differential equations take the form

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}(x^1)^2} = f^i\left(x^1, x^j, \frac{\mathrm{d}x^j}{\mathrm{d}x^1}\right), \quad i, j = 2, 3, \ldots, m.$$

In other words, there is always locally a member of the projective class for which $\Gamma^1 = 0$; then $f^i(x^a, y^j) = -2\Gamma^i(x^a, 1, y^j)$. Conversely, given a system of $m - 1$ second-order differential equations in the variables x^i, with parameter x^1, we can locally recover a spray by setting

$$\Gamma^1 = 0, \quad \Gamma^i(x^a, u^a) = -\frac{1}{2}(u^1)^2 f^i(x^a, u^j/u^1).$$

Such a spray is reversible.

If we make a point transformation (a coordinate transformation involving all of the coordinates x^a) the spray corresponding to the new system of differential equations will not be the same as that corresponding to the original one; but it will be projectively equivalent to it. The invariants of the system of second-order ordinary differential equations under point transformations will be the projective invariants of the corresponding projective equivalence class of sprays.

It will be useful to be able to represent the projective quantities in terms of the f^i. We therefore compute the fundamental invariants Π^a_{bc} of the spray

$$u^a\frac{\partial}{\partial x^a} - 2\Gamma^a\frac{\partial}{\partial u^a}, \quad \Gamma^1 = 0, \quad \Gamma^i\left(x^a, u^a\right) = -\frac{1}{2}\left(u^1\right)^2 f^i\left(x^a, u^j/u^1\right).$$

We set

$$\gamma^i_j = -\frac{1}{2}\frac{\partial f^i}{\partial y^j}, \quad \gamma^i_{jk} = \frac{\partial \gamma^i_j}{\partial y^k}, \quad \gamma^i_{jkl} = \frac{\partial \gamma^i_{jk}}{\partial y^l},$$
$$\gamma = \gamma^k_k, \quad \gamma_i = \frac{\partial \gamma}{\partial y^i} = \gamma^k_{ik}, \quad \gamma_{ij} = \frac{\partial^2 \gamma}{\partial y^i \partial y^j}.$$

We denote by $\mathrm{d}/\mathrm{d}x^1$ the differential operator

$$\frac{\mathrm{d}}{\mathrm{d}x^1} = \frac{\partial}{\partial x^1} + y^i \frac{\partial}{\partial x^i} + f^i \frac{\partial}{\partial y^i}$$

(it is a vector field on a jet bundle, the second-order differential equation field or SODE of the system of equations $\ddot{x}^i = f^i$), and set

$$\Phi^i_j = \frac{\partial f^i}{\partial x^j} + \frac{\mathrm{d}}{\mathrm{d}x^1}\left(\gamma^i_j\right) + \gamma^i_k \gamma^k_j;$$

Φ^i_j is called in the relevant literature, with an unfortunate disagreement over sign, the Jacobi endomorphism of the second-order differential equation field [6]. Then γ^i_{jkl} and Φ^i_j are invariants of the second-order differential equation field under the restricted class of coordinate transformations $\hat{x}^1 = x^1$, $\hat{x}^i = \hat{x}^i(x^1, x^i)$, that is, transformations that preserve the parametrization.

We will show that for $m > 2$ the Douglas tensor D^a_{bcd} and the Weyl tensor W^a_b are completely determined by the quantities

$$K^i_{jkl} = \gamma^i_{jkl} - \frac{1}{m+1}\left(\delta^i_j \gamma_{kl} + \delta^i_k \gamma_{jl} + \delta^i_l \gamma_{jk}\right), \quad L^i_j = \Phi^i_j - \frac{1}{(m-1)}\delta^i_j \Phi^k_k.$$

Since the projective curvature tensor P^a_{bcd} determines and is determined by the Weyl tensor, it too is completely determined by these quantities. This is related to a result of Fels [11], who showed, using Cartan's method of equivalence, that K^i_{jkl} and L^i_j are the fundamental invariants of the system of second-order ordinary differential equations under point transformations. In fact the point transformation invariants of a system of second-order differential equations, or equivalently the projective invariants of the corresponding spray, are completely determined by the trace-free parts of the invariants of the system under parametrization-preserving transformations.

To derive this result we need to compute several quantities from the Γ^a by differentiating with respect to the u^a and taking traces. The calculations are much simplified by the fact that the quantities involved are homogeneous of various degrees in the u^a. Any function ϕ of degree n is determined by its value at $u^1 = 1$, since $\phi(u^a) = (u^1)^n \phi(1, u^i/u^1)$ (this is of course just the principle used to define the spray coefficients). Moreover, we have Euler's theorem at our disposal. In the formulae below we assume that $u^1 \neq 0$, and write y^i for u^i/u^1.

First we have

$$\Gamma^1_b = 0, \quad \Gamma^i_j = u^1 \gamma^i_j, \quad \Gamma^i_1 = -(u^1)(f^i + y^l \gamma^i_l).$$

For the Γ^a_{bc} we obtain

$$\Gamma^1_{bc} = 0, \quad \Gamma^i_{jk} = \gamma^i_{jk}, \quad \Gamma^i_{1j} = \Gamma^i_{j1} = \gamma^i_j - y^l \gamma^i_{jl}, \quad \Gamma^i_{11} = -f^i - 2y^l \gamma^i_l + y^l y^m \gamma^i_{lm}.$$

Next, the traces:

$$\Gamma = u^1 \gamma; \quad \Gamma_i = \gamma_i, \quad \Gamma_1 = \gamma - y^l \gamma_l;$$

and their derivatives

$$\Gamma_{ij} = \left(u^1\right)^{-1} \gamma_{ij}, \quad \Gamma_{1i} = \Gamma_{i1} = -\left(u^1\right)^{-1} y^l \gamma_{il} \quad \Gamma_{11} = \left(u^1\right)^{-1} y^l y^m \gamma_{lm}.$$

It follows that the fundamental invariants Π^a_{bc} are given in terms of the f^i and their derivatives by

$$\Pi^1_{11} = -\frac{1}{m+1}\left(2\gamma - 2y^l \gamma_l + y^l y^m \gamma_{lm}\right)$$

$$\Pi^1_{1i} = -\frac{1}{m+1}\left(\gamma_i - y^l\gamma_{il}\right)$$
$$\Pi^1_{ij} = -\frac{1}{m+1}\gamma_{ij}$$
$$\Pi^i_{11} = -f^i - 2y^l\gamma^i_l + y^l y^m\gamma^i_{lm} - \frac{1}{m+1}y^i y^l y^m\gamma_{lm}$$
$$\Pi^i_{1j} = \gamma^i_j - y^l\gamma^i_{jl} - \frac{1}{m+1}\left(\left(\gamma - y^l\gamma_l\right)\delta^i_j - y^i y^l\gamma_{jl}\right)$$
$$\Pi^i_{jk} = \gamma^i_{jk} - \frac{1}{m+1}\left(\gamma_j\delta^i_k + \gamma_k\delta^i_j + y^i\gamma_{jk}\right).$$

Thus with $u^1 = 1$,

$$D^i_{jkl} = \gamma^i_{jkl} - \frac{1}{m+1}\left(\gamma_{jl}\delta^i_k + \gamma_{kl}\delta^i_j + \gamma_{jk}\delta^i_l + y^i\gamma_{jkl}\right).$$

We differentiate K^i_{jkl} with respect to y^m to obtain

$$\frac{\partial K^i_{jkl}}{\partial y^m} = \gamma^i_{jklm} - \frac{1}{m+1}\left(\delta^i_j\gamma_{klm} + \delta^i_k\gamma_{jlm} + \delta^i_l\gamma_{jkm}\right),$$

then take a trace to get

$$\frac{m-2}{m+1}\gamma_{jkl} = \frac{\partial K^m_{jkl}}{\partial y^m},$$

whence

$$D^i_{jkl} = K^i_{jkl} - \frac{1}{m-2}y^i\frac{\partial K^m_{jkl}}{\partial y^m}.$$

Furthermore

$$D^1_{jkl} = -\frac{1}{m+1}\gamma_{jkl} = -\frac{1}{m-2}\frac{\partial K^m_{jkl}}{\partial y^m}.$$

Now $u^d D^a_{bcd} = 0$, whence $D^a_{bc1} = -y^i D^a_{bci}$, so that the remaining components of D^a_{bcd} are determined by those which have already been calculated.

For the Jacobi endomorphism of the spray we have $R^1_a = 0$,

$$R^i_j = -\left(u^1\right)^2\left(\frac{\partial f^i}{\partial x^j} + \frac{\mathrm{d}}{\mathrm{d}x^1}\left(\gamma^i_j\right) + \gamma^i_k\gamma^k_j\right) = -\left(u^1\right)^2\Phi^i_j,$$

so that $R = -(u^1)^2\Phi^k_k$. Thus

$$R^i_j - \frac{1}{m-1}R\delta^i_j = -\left(u^1\right)^2 L^i_j, \quad R^1_j - \frac{1}{m-1}R\delta^1_j = 0.$$

Thus when $u^1 = 1$,

$$W^i_j = -L^i_j + \frac{1}{m+1}y^i\frac{\partial L^k_j}{\partial y^k}, \quad W^1_j = \frac{1}{m+1}\frac{\partial L^k_j}{\partial y^k}.$$

As before, the remaining components of W^a_b are determined by these, since $W^a_b u^b = 0$.

It follows that $L^i_j = 0$ is a necessary and sufficient condition for the spray to be isotropic.

3. The Thomas–Whitehead theory

We now discuss the Thomas–Whitehead theory. In its original version, and also in the version of Roberts [21] on which our account is based, this is a theory of projective equivalence classes of affine connections, so the first part of this section deals with the affine case. We show how to generalize the theory to arbitrary sprays in the final subsection.

The fundamental example is projective space P^m itself. We begin by considering connections on projective space, as motivation for the general constructions.

3.1. Connections on projective space

As a manifold, P^m is the quotient of $\mathbf{R}^{m+1}-\{0\}$ under the multiplicative action of $\mathbf{R}-\{0\}$; the infinitesimal generator of this action is the radial vector field given in Cartesian coordinates by $x^\alpha \partial_\alpha = \Upsilon$. We may represent objects on P^m as objects on $\mathbf{R}^{m+1} - \{0\}$ transforming appropriately under the action; for convenience this will be expressed in terms of the Lie derivative with respect to Υ, together with invariance under the reflection map $j : x \mapsto -x$. So functions on P^m may be represented by functions f on $\mathbf{R}^{m+1} - \{0\}$ satisfying $\Upsilon f = 0$ and $j^*(f) = f$: call the set of such functions $\mathcal{F}_\Upsilon$. Similarly, vector fields on P^m may be represented by equivalence classes of vector fields X on $\mathbf{R}^{m+1} - \{0\}$ satisfying $\mathcal{L}_\Upsilon X \propto \Upsilon$ and $j_* X = X$, with equivalence $Y \equiv X$ if $Y - X \propto \Upsilon$. Let $\mathfrak{X}_\Upsilon$ denote the set of such vector fields, and for $X \in \mathfrak{X}_\Upsilon$ let $[\![X]\!]$ denote the equivalence class of X. The set $[\![\mathfrak{X}_\Upsilon]\!]$ of equivalence classes $[\![X]\!]$ for $X \in \mathfrak{X}_\Upsilon$ is a Lie algebra over the module $\mathcal{F}_\Upsilon$, with $[\, [\![X]\!] , [\![Y]\!] \,] = [\![\, [X, Y] \,]\!]$. Furthermore, for any $f \in \mathcal{F}_\Upsilon$, $Xf \in \mathcal{F}_\Upsilon$ if $X \in \mathfrak{X}_\Upsilon$ and $Yf = Xf$ if $Y \equiv X$: thus $[\![X]\!]f$ is well-defined (as Xf); $[\![\mathfrak{X}_\Upsilon]\!]$ acts as derivations on $\mathcal{F}_\Upsilon$; and the Lie bracket of equivalence classes is the commutator of the corresponding derivations.

We will define a covariant derivative operator on $[\![\mathfrak{X}_\Upsilon]\!]$ as a map $\nabla : [\![\mathfrak{X}_\Upsilon]\!] \times [\![\mathfrak{X}_\Upsilon]\!] \to [\![\mathfrak{X}_\Upsilon]\!]$ which is $\mathbf{R}$-bilinear, $\mathcal{F}_\Upsilon$-linear in the first variable, and satisfies

$$\nabla_{[\![X]\!]}(f[\![Y]\!]) = f\nabla_{[\![X]\!]}[\![Y]\!] + ([\![X]\!]f)[Y].$$

A covariant derivative is symmetric if

$$\nabla_{[\![X]\!]}[\![Y]\!] - \nabla_{[\![Y]\!]}[\![X]\!] = [\, [\![X]\!] , [\![Y]\!] \,].$$

We now relate such operators to the standard covariant derivative D on $\mathbf{R}^{m+1}$, by the device of choosing a representative of each equivalence class. Let ϑ be a 1-form on $\mathbf{R}^{m+1} - \{0\}$ such that $\langle \Upsilon, \vartheta \rangle = 1$ and $j^*\vartheta = \vartheta$, and for any vector field X set $\tilde{X} = X - \langle X, \vartheta \rangle \Upsilon$. Then if $Y \equiv X$, $\tilde{Y} = \tilde{X}$; and if $X \in \mathfrak{X}_\Upsilon$, $\tilde{X} \in \mathfrak{X}_\Upsilon$ also. Thus such a 1-form ϑ enables one to select a representative of each equivalence class, in fact by the condition $\langle X, \vartheta \rangle = 0$. If, furthermore, $\mathcal{L}_\Upsilon \vartheta = 0$ then $\mathcal{L}_\Upsilon \tilde{X} = 0$. Now Υ is an infinitesimal affine transformation of D, and so when $\mathcal{L}_\Upsilon \tilde{X} = \mathcal{L}_\Upsilon \tilde{Y} = 0$

$$\mathcal{L}_\Upsilon \left(D_{\tilde{X}} \tilde{Y} \right) = D_{\mathcal{L}_\Upsilon \tilde{X}} \tilde{Y} + D_{\tilde{X}} \left(\mathcal{L}_\Upsilon \tilde{Y} \right) = 0$$

also. Furthermore, j is an affine transformation, so when $j_* \tilde{X} = \tilde{X}$ and $j_* \tilde{Y} = \tilde{Y}$, $j_*(D_{\tilde{X}} \tilde{Y}) = D_{\tilde{X}} \tilde{Y}$. So for any choice of ϑ such that $\langle \Upsilon, \vartheta \rangle = 1$, $\mathcal{L}_\Upsilon \vartheta = 0$ and $j^*\vartheta = \vartheta$ we may set

$$\nabla^\vartheta_{[\![X]\!]}[\![Y]\!] = \left[\!\left[D_{\tilde{X}} \tilde{Y} \right]\!\right];$$

then ∇^ϑ is a symmetric connection on $\mathfrak{X}_\Upsilon$.

We may now consider the geodesics of ∇^ϑ. First of all, a geodesic will be a 2-surface Σ in $\mathbf{R}^{m+1} - \{0\}$ invariant under the action generated by Υ, that is, ruled by radial lines. It will be defined by any curve in it transverse to the radial lines, and among such curves we can choose those σ whose tangent vectors satisfy $\langle \dot\sigma, \vartheta \rangle = 0$. Such curves are mapped to each other by the action generated by Υ, so it is enough to consider one of them. Then Σ will be a geodesic of ∇^ϑ if and only if $[\![D_{\dot\sigma} \dot\sigma]\!] \propto [\![\dot\sigma]\!]$, that is, if and only if $D_{\dot\sigma} \dot\sigma = \ddot\sigma$ is a linear combination of $\dot\sigma$ and $\Upsilon|_\sigma$. But this means that the tangent planes to Σ at all points on it are parallel to one another, and therefore that Σ is itself a plane. Thus the geodesics of ∇^ϑ are the straight lines in P^m. We note for future reference that Υ has the property that $D\Upsilon = \mathrm{id}$, where id is the identity tensor; and indeed this determines Υ up to the addition of a constant vector field.

We can describe the idea behind the construction of Roberts as follows: to introduce for any manifold M, a manifold $\mathcal{V}M$ of one higher dimension, whose role in relation to M is to be like that of $\mathbf{R}^{m+1} - \{0\}$ in relation to P^m; and on

$\mathcal{V}M$ to define a covariant derivative operator whose role in relation to a projective equivalence class of connections on M is to be like that of D in relation to P^m as described above. We can motivate the construction of $\mathcal{V}M$ by introducing a particular way of thinking about $\mathbf{R}^{m+1} - \{0\}$ in this context.

Let Ω be the standard volume form on $\mathbf{R}^{m+1}$. Let π be the projection $\mathbf{R}^{m+1} \to \mathrm{S}^m$, where S^m, the m-sphere, is the quotient of $\mathbf{R}^{m+1} - \{0\}$ by the action generated by Υ, so that P^m is obtained from S^m by identifying diametrically opposite points. Then for any point $p \in \mathbf{R}^{m+1}$, $p \neq 0$, we can define an m-covector θ at $\pi(p) \in \mathrm{S}^m$ as follows: let ξ_a be any m elements of $T_{\pi(p)}\mathrm{S}^m$, and let v_a be any m elements of $T_p\mathbf{R}^{m+1}$ such that $\pi_{p*}v_a = \xi_a$; set

$$\theta(\xi_1, \xi_2, \ldots, \xi_m) = \Omega_p(\Upsilon_p, v_1, v_2, \ldots, v_m);$$

θ is clearly well-defined since adding a multiple of Υ_p to any v_a doesn't change the value of the right-hand side. Now take any $s \in \mathbf{R}$, $s > 0$, and carry out the same construction but starting at sp. It is clear that the right-hand side gets multiplied by s^{m+1}. There is therefore a map $\varphi : \mathbf{R}^{m+1} - \{0\} \to \bigwedge^m \mathrm{S}^m$ such that $\varphi(sp) = s^{m+1}\varphi(p)$. In this case φ will be a diffeomorphism of $\mathbf{R}^{m+1} - \{0\}$ with either of the two classes of oriented volume forms on S^m. There is no need to take this any further here: our aim was just to suggest that it will be profitable to consider volume forms.

A somewhat similar account is to be found in [15], but as an application of Roberts's construction rather than as motivation for it.

3.2. The volume bundle

The basic idea of the Thomas–Whitehead theory of projectively equivalent connections is to represent a projective equivalence class on an m-dimensional manifold by a single connection on a manifold of dimension $m+1$, extending the approach of the previous subsection from projective space to a more general manifold M. We start by describing the appropriate $(m+1)$-dimensional manifold, broadly following Roberts [21] but diverging from him over some details.

We start with the non-zero volume elements $\theta \in \bigwedge^m T^*M$; the set of pairs $[\pm\theta]$ of such elements will be called the volume bundle of M (strictly speaking it should be called the unoriented volume bundle but we will normally omit the prefix 'unoriented') and denoted by $\mathcal{V}M$. It is indeed a bundle, with projection $\nu : \mathcal{V}M \to M$, defined by $\nu[\pm\theta] = x$ whenever $\theta, -\theta \in \bigwedge^m T_x^*M$. If x^a are coordinates on M then a candidate for the fibre coordinate on the (one-dimensional) fibre of ν is $|v|$, where v satisfies

$$\theta = v(\theta)\left(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m\right)_x$$

for any $\theta \in \bigwedge^m T^*M$; however, in view of the discussion in the previous subsection we choose instead to use $x^0 = |v|^{1/(m+1)}$ as the fibre coordinate, with the convention that the positive root is to be taken if m is odd so that $x^0 > 0$. The local trivializations defined in this way describe a principal $\mathbf{R}_+$-bundle structure on ν ($\mathbf{R}_+$ is the multiplicative group of positive reals). We will let $\mu : \mathcal{V}M \times \mathbf{R}_+ \to \mathcal{V}M$ denote the corresponding (right) action $[\pm\theta] \mapsto [\pm s^{m+1}\theta]$ of $\mathbf{R}_+$ on the fibres of ν, and also write $\mu_s : \mathcal{V}M \to \mathcal{V}M$ for the map defined by $\mu_s([\pm\theta]) = \mu([\pm\theta], s)$. The fundamental vector field of this bundle, which corresponds to the radial vector field of the previous sub-section, will be denoted by Υ; in coordinates

$$\Upsilon = x^0 \frac{\partial}{\partial x^0}.$$

Although our construction of the volume bundle is similar to that used by Roberts [21], it is not quite the same. A small difference is that Roberts uses the structure of an $\mathbf{R}$-bundle rather than an $\mathbf{R}_+$-bundle, by exploiting the exponential isomorphism. More significant is that his bundle is built from m-vectors rather than m-covectors—the two bundles are isomorphic, but the natural fibre coordinate is $|v|^{-1}$ rather than $|v|$. Another significant difference is that our $\mathbf{R}_+$-bundle structure uses multiplication by s^{m+1} rather than multiplication by s as the right action, and so our fundamental vector field Υ is $-(m+1)$ times the one used by Roberts.

The volume bundle has some additional natural structure, a so-called odd scalar density, which is defined in the following way. Observe first that $\bigwedge^m T^*M$, as a bundle of m-covectors, has a tautological m-form Θ; in coordinates $\Theta = v\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$. The differential $\mathrm{d}\Theta$ is a natural volume form on $\bigwedge^m T^*M$ defining, at each point $[\pm\theta] \in \mathcal{V}M$, a pair of $(m+1)$-covectors differing only in sign; this is the odd scalar density we require. We will denote it by $|\mathrm{d}\Theta|$.

In the coordinates on $\mathcal{V}M$, this may be written as

$$\pm(m+1)\left(x^0\right)^m \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m.$$

3.3. *TW*-connections

Roberts's version of the Thomas–Whitehead theory is based on his notion of a Thomas–Whitehead projective connection, or *TW*-connection for short. A *TW*-connection is a symmetric affine connection $\tilde{\nabla}$ on the volume bundle $\mathcal{V}M$ which is invariant under the $\mathbf{R}_+$ action on $\nu : \mathcal{V}M \to M$ and which satisfies the condition that $\tilde{\nabla}\Upsilon = \mathrm{id}$, where id is the identity tensor on $\mathcal{V}M$. The invariance condition is equivalent to saying that Υ is an infinitesimal affine transformation of $\tilde{\nabla}$, and we will generally use it in this form. (The definition above is essentially the one given in [21], with the difference that the formulae there differ from ours by the constant factor $-(m+1)$ as we use a different fundamental vector field.) These conditions on $\tilde{\nabla}$, when expressed in terms of its connection coefficients $\tilde{\Gamma}^\gamma_{\alpha\beta}$ with respect to coordinates (x^α), adapted to the bundle structure, give $\tilde{\Gamma}^0_{a0} = \tilde{\Gamma}^\gamma_{00} = 0$, $\tilde{\Gamma}^b_{a0} = (x^0)^{-1}\delta^b_a$; furthermore, the $\tilde{\Gamma}^c_{ab}$ are functions on M transforming as the components of the fundamental descriptive invariant of a restricted path space (though not necessarily satisfying $\tilde{\Gamma}^a_{ab} = 0$), while the $\tilde{\Gamma}^0_{ab}$ are of the form $x^0\alpha_{ab}$ where the α_{ab} are functions on M, transforming appropriately. There is therefore a many–one correspondence between *TW*-connections and restricted path spaces. The geodesic equations for a *TW*-connection are

$$\ddot{x}^c + \tilde{\Gamma}^c_{ab}\dot{x}^a\dot{x}^b = -2\left(\dot{x}^0/x^0\right)\dot{x}^c, \quad \ddot{x}^0 + x^0\,\alpha_{ab}\dot{x}^a\dot{x}^b = 0.$$

The first of these defines the paths on M. The second equation tells us that the terms α_{ab} in the connection essentially determine a preferred parametrization of the paths. Suppose that we wish to make a change of parametrization so that with respect to the new parameter the equations become

$$\ddot{x}^c + \tilde{\Gamma}^c_{ab}\dot{x}^a\dot{x}^b = 0.$$

Then from the first equation s must satisfy $\ddot{s} = -2(\dot{x}^0/x^0)\dot{s}$, and from the second

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\ddot{s}}{\dot{s}}\right) - \frac{1}{2}\left(\frac{\ddot{s}}{\dot{s}}\right)^2 = 2\alpha_{ab}\dot{x}^a\dot{x}^b.$$

The left-hand side of this equation is the Schwarzian derivative of s, sometimes denoted by $\mathrm{S}(s)$. It is known that if f is a Möbius function of t,

$$f(t) = \frac{at+b}{ct+d}$$

for some constants a, b, c and d with $ad - bc \neq 0$, then $\mathrm{S}(f \circ s) = \mathrm{S}(s)$; thus if s is a reparametrization to the new parameter so is $f \circ s$ for any Möbius function f.

The importance of a *TW*-connection on the volume bundle is that it gives rise to a family of connections on M. It is shown in [21] that given a *TW*-connection $\tilde{\nabla}$, with the aid of any 1-form ϑ on $\mathcal{V}M$ which is $\mathbf{R}_+$-invariant and satisfies $\langle \Upsilon, \vartheta \rangle = 1$ one can construct a symmetric affine connection ∇^ϑ on M whose geodesics are the paths of the restricted path space corresponding to $\tilde{\nabla}$, just as we showed for P^m earlier. Such a 1-form ϑ is the connection form of a connection on the principal bundle $\mathcal{V}M \to M$. It is well known (see for example [17] Chapter II, Theorem 2.1) that every principal bundle over a paracompact manifold admits a global connection. It follows that every restricted path space on a paracompact manifold M is the space of geodesic paths of some symmetric affine connection on M.

In fact $\tilde{\nabla}$ gives rise in this way to a projective equivalence class $[\nabla]$ of symmetric affine connections on M, the different members of the class corresponding to different choices of ϑ; the difference $\vartheta' - \vartheta$ of two such 1-forms on $\mathcal{V}M$ is the pull-back of a 1-form on M, which determines the projective transformation relating the two corresponding connections on M. Conversely, each such projective equivalence class $[\nabla]$ gives rise to many *TW*-connections, and in particular to a unique *TW*-connection $\tilde{\nabla}$ satisfying the additional conditions that $\tilde{\nabla}(|\mathrm{d}\Theta|) = 0$ and that the Ricci curvature of $\tilde{\nabla}$ vanishes. In coordinates,

$$\tilde{\nabla}_0(\partial_0) = 0, \quad \tilde{\nabla}_0(\partial_b) = \tilde{\nabla}_b(\partial_0) = \left(x^0\right)^{-1}\partial_b, \quad \tilde{\nabla}_a(\partial_b) = \Pi^c_{ab}\partial_c - \frac{1}{m-1}x^0\mathfrak{R}_{ab}\partial_0,$$

where as before Π^c_{ab} is the fundamental descriptive invariant of the equivalence class $[\nabla]$ and $\mathfrak{R}_{ab}$ its Ricci 'tensor'. More generally, those *TW*-connections for which $\tilde\nabla(|\mathrm{d}\Theta|) = 0$ take the form

$$\tilde\nabla_0(\partial_0) = 0, \quad \tilde\nabla_0(\partial_b) = \tilde\nabla_b(\partial_0) = \left(x^0\right)^{-1}\partial_b, \quad \tilde\nabla_a(\partial_b) = \Pi^c_{ab}\partial_c - x^0\alpha_{ab}\partial_0.$$

That is to say, the condition $\tilde\nabla(|\mathrm{d}\Theta|) = 0$ forces $\tilde\Gamma^c_{ab}$ to be Π^c_{ab}, that is, to have vanishing trace. We will accordingly call such a *TW*-connection trace-free, and we will call the trace-free *TW*-connection whose Ricci curvature vanishes the normal *TW*-connection for the given projective equivalence class of affine connections.

It is also the case that if M is paracompact, $\nu : \mathcal{V}M \to M$ admits global sections ([17] Chapter I, Theorem 5.7). A global section σ determines a connection on the principal bundle, which is integrable, and whose connection 1-form is exact, say $\mathrm{d}\varpi$; the function ϖ satisfies $\Upsilon\varpi = 1$, and the horizontal submanifolds are the level sets of ϖ. Such a global section is called a choice of projective scale by Bailey et al. in [1]. The corresponding affine connection $\nabla^{\mathrm{d}\varpi}$ has the property that its Ricci tensor is symmetric, and any connection in the projective equivalence class with this property is determined in this way. The projective transformation relating two connections with this property is given by an exact 1-form on M.

We can also relate this construction to that of the so-called tractor bundle introduced in [1]. Suppose given a connection form ϑ; let H_a be the corresponding horizontal lifts of the ∂_a from M to $\mathcal{V}M$. The invariance of the connection form implies that $\mathcal{L}_\Upsilon H_a = 0$. Now consider the vector fields X on $\mathcal{V}M$ such that $\tilde\nabla_\Upsilon X = 0$; call them Υ-vectors. We may equivalently write the defining condition as $\mathcal{L}_\Upsilon X = -X$. The Υ-vectors form a module over functions on M. For any Υ-vector X and for any Y such that $\mathcal{L}_\Upsilon Y \propto \Upsilon$ (i.e. any projectable vector field Y), $\tilde\nabla_Y X$ is also a Υ-vector, by virtue of the rules for a *TW*-connection. If X is a Υ-vector and ϑ is a connection form then the horizontal component of X, that is, $X - \langle X, \vartheta\rangle\Upsilon$, is also a Υ-vector, as is its vertical component $\langle X, \vartheta\rangle\Upsilon$. We can write a vertical Υ-vector as $\mu^0\Upsilon$; then $\Upsilon\mu^0 = -\mu^0$, from which it follows that μ^0 is a scalar density on M of weight $-1/(m+1)$. We can write a horizontal Υ-vector as $\mu^a H_a$; then similarly the μ^a are components of a contravariant vector density of weight $-1/(m+1)$.

We can now define a covariant derivative operator on Υ-vectors, with respect to vector fields on M, by restricting the arguments of the *TW*-connection to be respectively projectable vector fields and Υ-vectors. For a trace-free *TW*-connection the representation of this covariant derivative with respect to an exact connection form $\vartheta = \mathrm{d}\varphi$ coincides with the formulae given in [1].

3.4. TW-connections and affine sprays

A symmetric affine connection determines and is determined by its corresponding affine spray; it is therefore not surprising that we can specify *TW*-connections, and in particular the normal *TW*-connection, entirely in terms of sprays.

To a *TW*-connection there corresponds an affine spray on $T(\mathcal{V}M)$, say $\tilde S$. The defining conditions for a *TW*-connection, when expressed in terms of $\tilde S$, turn out to be

$$\mathcal{L}_{\Upsilon^{\mathrm{C}}}\tilde S = 0; \quad \mathcal{L}_{\Upsilon^{\mathrm{V}}}\tilde S = \Upsilon^{\mathrm{C}} - 2\tilde\Delta$$

where Υ^{C} and Υ^{V} are respectively the complete and vertical lifts of Υ to $T(\mathcal{V}M)$, and $\tilde\Delta$ is the Liouville field on $T(\mathcal{V}M)$. The first of these conditions is equivalent to the requirement that Υ is an infinitesimal affine transformation of the *TW*-connection, and the second to the requirement that $\tilde\nabla\Upsilon = \mathrm{id}$. The second condition may be reformulated in terms of the horizontal lift Υ^{H} of Υ to $T(\mathcal{V}M)$: since for any vector field X on $\mathcal{V}M$, $X^{\mathrm{H}} = \frac{1}{2}(\mathcal{L}_{X^{\mathrm{V}}}\tilde S + X^{\mathrm{C}})$, we have $\Upsilon^{\mathrm{C}} - \Upsilon^{\mathrm{H}} = \tilde\Delta$. A variant of this formula will be important later.

Both of the claims above are easily confirmed by the following general coordinate calculations. Consider a manifold ($\mathcal{V}M$ for example) equipped with a symmetric affine connection ∇ and corresponding affine spray S. The condition in coordinates (x^α) for a vector field X to be an affine transformation of ∇ is

$$\frac{\partial^2 X^\gamma}{\partial x^\alpha \partial x^\beta} + \frac{\partial X^\delta}{\partial x^\alpha}\Gamma^\gamma_{\delta\beta} + \frac{\partial X^\delta}{\partial x^\beta}\Gamma^\gamma_{\delta\alpha} - \frac{\partial X^\gamma}{\partial x^\delta}\Gamma^\delta_{\alpha\beta} + X^\delta\frac{\partial \Gamma^\gamma_{\alpha\beta}}{\partial x^\delta} = 0,$$

while the condition that $\mathcal{L}_{X^{\mathrm{C}}} S = 0$ is just this contracted with u^α and u^β. The condition that $\nabla X = \mathrm{id}$ is

$$\frac{\partial X^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma} X^\gamma = \delta^\alpha_\beta,$$

while a short calculation gives

$$\mathcal{L}_{X^{\mathrm{V}}} S = X^{\mathrm{C}} - 2u^\beta \left(\frac{\partial X^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma} X^\gamma \right) \frac{\partial}{\partial u^\alpha}.$$

The odd scalar density on $\mathcal{V}M$ defines a volume form vol on $T(\mathcal{V}M)$ by 'squaring' (and ignoring a constant factor):

$$\mathrm{vol} = \left(x^0\right)^{2m} \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m \wedge \mathrm{d}u^0 \wedge \mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^m.$$

It is easy to see that the necessary and sufficient condition for a *TW*-connection to be trace-free is that the corresponding spray satisfies $\mathcal{L}_{\tilde{S}} \mathrm{vol} = 0$.

For any affine spray S, the vertical component of $\mathcal{L}_S X^{\mathrm{H}}$ is

$$R^\alpha_{\beta\gamma\delta} u^\beta X^\gamma u^\delta \frac{\partial}{\partial u^\alpha}$$

where $R^\alpha_{\beta\gamma\delta}$ is the curvature of the corresponding connection; the quantity $R^\alpha_{\beta\gamma\delta} u^\beta u^\delta$, which is a type (1, 1) tensor field along the projection $\tau_M : TM \to M$ in component form, is the Jacobi endomorphism of the spray. The trace of the Jacobi endomorphism is just $R_{\beta\delta} u^\beta u^\delta$, a function on TM formed out of the Ricci curvature of the connection. In this way the Ricci curvature can be expressed entirely in terms of the spray. Then a trace-free *TW*-connection is the normal *TW*-connection if and only if the trace of its Jacobi endomorphism vanishes.

It can be shown that given a projective equivalence class of affine sprays on a manifold M there is a unique affine spray $\tilde{S}$ on $T(\mathcal{V}M)$, whose integral curves when projected into M belong to the path space determined by the projective class, such that

- $\mathcal{L}_{\Upsilon^{\mathrm{C}}} \tilde{S} = 0$;
- $\Upsilon^{\mathrm{C}} - \Upsilon^{\mathrm{H}} = \tilde{\Delta}$;
- $\mathcal{L}_{\tilde{S}} \mathrm{vol} = 0$;
- the Jacobi endomorphism of $\tilde{S}$ has vanishing trace.

This spray is given in coordinates adapted to $\mathcal{V}M$ by

$$\tilde{S} = u^\alpha \frac{\partial}{\partial x^\alpha} - \left(\Pi^a_{bc} u^b u^c + \left(x^0\right)^{-1} u^0 u^a \right) \frac{\partial}{\partial u^a} + \frac{1}{(m-1)} x^0 \mathfrak{R}_{cd} u^c u^d \frac{\partial}{\partial u^0}.$$

We call this the normal *TW*-spray; we can define the normal *TW*-connection as the symmetric affine connection determined by the normal *TW*-spray.

The relation between the normal *TW*-spray and the underlying projective class of affine sprays can be conveniently described in terms of a new bundle, by taking advantage of the fact that by definition the *TW*-spray $\tilde{S}$ satisfies $\mathcal{L}_{\Upsilon^{\mathrm{C}}} \tilde{S} = 0$. Consider the tangent bundle to the volume bundle, $\tau_{\mathcal{V}M} : T(\mathcal{V}M) \to \mathcal{V}M$. The action μ_s of $\mathbf{R}_+$ on the fibres of $\nu : \mathcal{V}M \to M$ lifts to a linear action $\mu_{s*} : T(\mathcal{V}M) \to T(\mathcal{V}M)$, which is just the derivative of μ_s. Let $\mathcal{W}M$ be the space of orbits of μ_*; then $\mathcal{W}M$ is a vector bundle over M with $m+1$-dimensional fibres, and with coordinates (x^a, u^a, w) where $w = (x^0)^{-1} u^0$. It is also a line bundle over TM, whose fibres, the orbits of μ_*, are the integral curves of Υ^{C}.

Since the *TW*-spray $\tilde{S}$ satisfies the condition $\mathcal{L}_{\Upsilon^{\mathrm{C}}} \tilde{S} = 0$, it projects to a vector field on $\mathcal{W}M$, say $\tilde{S}_{\mathcal{W}}$. We can use $\tilde{S}_{\mathcal{W}}$ to construct the projective equivalence class of affine sprays on TM corresponding to the *TW*-spray, as follows. The line bundle $\rho : \mathcal{W}M \to TM$ admits global sections. A section σ is linear (in the fibre coordinates of $TM \to M$) if $\sigma_*(\Delta) = \Delta_{\mathcal{W}} \circ \sigma$, where Δ is the Liouville field of TM and $\Delta_{\mathcal{W}}$ is that of the vector bundle $\tau : \mathcal{W}M \to M$. For any linear section σ, $\rho_*(\tilde{S}_{\mathcal{W}}|_\sigma)$ is a spray on M. The difference between two linear sections is a linear function on TM, so the corresponding sprays are projectively equivalent. The projective equivalence class so defined is the one that generates the *TW*-spray, and equivalently the *TW*-connection.

In terms of the construction given previously, a 1-form ϑ on $\mathcal{V}M$ defines a linear function $\hat{\vartheta}$ on $T(\mathcal{V}M)$; if $\mathcal{L}_{\Upsilon}\vartheta = 0$ then $\Upsilon^{\mathrm{C}}(\hat{\vartheta}) = 0$, in which case ϑ determines a function on $\mathcal{W}M$; the zero set of such a function defines a linear section σ of $\mathcal{W}M \to TM$, and the spray on TM determined by σ is the spray of the connection on M determined by ϑ.

3.5. BTW-connections

We have just shown that the geometry of a projective equivalence class of affine sprays on TM can be described in terms of a single affine spray on $T(\mathcal{V}M)$. We have so far followed the historical order of events, by developing the theory of *TW*-connections first, basing our account on the work of Roberts [21], and subsequently showing that the defining properties of a *TW*-connection can be specified in terms of its spray. We are now faced with the problem of generalizing these ideas to the case of a projective equivalence class of (not necessarily affine) sprays. The properties formerly used to define a *TW*-connection do not translate straightforwardly into properties of Berwald connections; however, the equivalent properties for an affine spray carry over almost without change to general sprays. We will therefore approach the definition of the Berwald connection on $T^\circ(\mathcal{V}M)$ which generalizes the normal *TW*-connection, which we will call the Berwald–Thomas–Whitehead projective connection, or *BTW*-connection for short (we will deal only with the analogue of the normal *TW*-connection and therefore need no qualifier), by first proving the existence of a uniquely determined spray on $T^\circ(\mathcal{V}M)$ which carries all the information about a given projective class of sprays on $T^\circ M$. We call this spray the *BTW*-spray, and define the *BTW*-connection as the Berwald connection of this spray.

For a spray $\tilde{S}$ on $T^\circ(\mathcal{V}M)$,

$$\tilde{S} = u^\alpha \frac{\partial}{\partial x^\alpha} - 2\tilde{\Gamma}^\alpha \frac{\partial}{\partial u^\alpha},$$

we denote by $\tilde{R}^\alpha_\beta$ its Jacobi endomorphism,

$$\tilde{R}^\alpha_\beta = 2\frac{\partial \tilde{\Gamma}^\alpha}{\partial x^\beta} - \tilde{S}\left(\tilde{\Gamma}^\alpha_\beta\right) - \tilde{\Gamma}^\alpha_\gamma \tilde{\Gamma}^\gamma_\beta,$$

and by $\tilde{R}$ the trace of $\tilde{R}^\alpha_\beta$. We again need the volume form vol on $T^\circ(\mathcal{V}M)$.

In the affine case the affine spray $\tilde{S}$ on $T(\mathcal{V}M)$ whose corresponding symmetric affine connection is the normal *TW*-connection of a given projective equivalence class of sprays on M is uniquely determined by the following conditions:

- $\mathcal{L}_{\Upsilon^{\mathrm{C}}}\tilde{S} = 0$;
- $\mathcal{L}_{\Upsilon^{\mathrm{V}}}\tilde{S} = \Upsilon^{\mathrm{C}} - 2\tilde{\Delta}$;
- $\mathcal{L}_{\tilde{S}}\mathrm{vol} = 0$;
- $\tilde{R} = 0$.

These conditions apply without change to general sprays on $T^\circ(\mathcal{V}M)$.

We will next derive the consequences of these conditions in terms of coordinates adapted to $\mathcal{V}M$, thus showing that sprays satisfying them exist locally; we postpone the proof of the global existence of such sprays until later. We find that

$$\mathcal{L}_{\Upsilon^{\mathrm{C}}}\tilde{S} = -2\left(x^0\frac{\partial \tilde{\Gamma}^\alpha}{\partial x^0} + u^0\frac{\partial \tilde{\Gamma}^\alpha}{\partial u^0}\right)\frac{\partial}{\partial u^\alpha} + 2\tilde{\Gamma}^0\frac{\partial}{\partial u^0},$$

$$\mathcal{L}_{\Upsilon^{\mathrm{V}}}\tilde{S} = \Upsilon^{\mathrm{C}} - 2x^0\frac{\partial \tilde{\Gamma}^\alpha}{\partial u^0}\frac{\partial}{\partial u^\alpha} - 2u^0\frac{\partial}{\partial u^0}.$$

So in order for $\tilde{S}$ to satisfy the first pair of conditions we must have

$$x^0\frac{\partial \tilde{\Gamma}^0}{\partial x^0} + u^0\frac{\partial \tilde{\Gamma}^0}{\partial u^0} = \tilde{\Gamma}^0, \quad x^0\frac{\partial \tilde{\Gamma}^a}{\partial x^0} + u^0\frac{\partial \tilde{\Gamma}^a}{\partial u^0} = 0, \quad \frac{\partial \tilde{\Gamma}^0}{\partial u^0} = 0, \quad x^0\frac{\partial \tilde{\Gamma}^a}{\partial u^0} = u^a.$$

It follows that

$$x^0 \frac{\partial \tilde{\Gamma}^0}{\partial x^0} = \tilde{\Gamma}^0,$$

whence $\tilde{\Gamma}^0 = x^0 G^0$ say, where G^0 is a function on $T^\circ M$ homogeneous of degree 2. Moreover

$$\frac{\partial \tilde{\Gamma}^a}{\partial x^0} = -\left(x^0\right)^{-2} u^0 u^a,$$

so that $\tilde{\Gamma}^a = (x^0)^{-1} u^0 u^a + G^a$, where again G^a is a function on $T^\circ M$ homogeneous of degree 2. Thus a spray satisfies the first pair of conditions if locally it takes the form

$$\tilde{S} = u^\alpha \frac{\partial}{\partial x^\alpha} - 2\left(G^a + \left(x^0\right)^{-1} u^0 u^a\right) \frac{\partial}{\partial u^a} - 2x^0 G^0 \frac{\partial}{\partial u^0}.$$

The remaining two conditions impose further restrictions on G^a and G^0. We have

$$\mathcal{L}_{\tilde{S}} \mathrm{vol} = \left(2m \frac{u^0}{x^0} - 2\left(G^a_a + m \frac{u^0}{x^0}\right)\right) \mathrm{vol},$$

so $\mathcal{L}_{\tilde{S}} \mathrm{vol} = 0$ if and only if $G^a_a = 0$. Now

$$\tilde{\Gamma}^0_0 = 0, \quad \tilde{\Gamma}^0_a = x^0 G^0_a, \quad \tilde{\Gamma}^a_0 = \frac{u^a}{x^0}, \quad \tilde{\Gamma}^a_b = G^a_b + \left(\frac{u^0}{x^0}\right) \delta^a_b,$$

whence $\tilde{\Gamma} = \tilde{\Gamma}^\alpha_\alpha = m u^0 / x^0$, and

$$\tilde{\Gamma}^\alpha_\beta \tilde{\Gamma}^\beta_\alpha = G^a_b G^b_a + 4G^0 + m \left(\frac{u^0}{x^0}\right)^2,$$

using $G^a_a = 0$ and the homogeneity of G^0. Thus

$$\begin{aligned}
\tilde{R} &= 2\left(\frac{\partial G^a}{\partial x^a} + G^0\right) + m\left(\left(\frac{u^0}{x^0}\right)^2 + 2G^0\right) - G^a_b G^b_a - 4G^0 - m\left(\frac{u^0}{x^0}\right)^2 \\
&= 2\frac{\partial G^a}{\partial x^a} - G^a_b G^b_a + 2(m-1)G^0,
\end{aligned}$$

so that the fourth condition is satisfied (given that the others are) if and only if

$$G^0 = -\frac{1}{2(m-1)} \left(2\frac{\partial G^a}{\partial x^a} - G^a_b G^b_a\right).$$

Now $\mathcal{L}_{\Upsilon^{\mathrm{C}}} \tilde{S} = 0$ is the necessary and sufficient condition for $\tilde{S}$ to project to a vector field on $\mathcal{W}^\circ M$, the restriction of $\mathcal{W}M$ to $T^\circ M$; call this vector field $\tilde{S}_{\mathcal{W}}$. With coordinate $w = u^0/x^0$ we have

$$u^0 \frac{\partial}{\partial x^0} \mapsto -w^2 \frac{\partial}{\partial w}, \quad x^0 \frac{\partial}{\partial u^0} \mapsto \frac{\partial}{\partial w},$$

so that

$$\tilde{S}_{\mathcal{W}} = u^a \frac{\partial}{\partial x^a} - 2(G^a + w u^a) \frac{\partial}{\partial u^a} - (w^2 + 2G^0) \frac{\partial}{\partial w},$$

with G^a, G^0 as above. Now $\rho : \mathcal{W}^\circ M \to T^\circ M$ is a line bundle. It admits global sections. A section σ is homogeneous if $\sigma_*(\Delta) = \Delta_{\mathcal{W}} \circ \sigma$, where Δ is the Liouville field of $T^\circ M$ and $\Delta_{\mathcal{W}}$ is that of the vector bundle $\tau : \mathcal{W}M \to M$ restricted to $\mathcal{W}^\circ M$. For any homogeneous section σ, $\rho_*(\tilde{S}_{\mathcal{W}}|_\sigma)$ is a spray on $T^\circ M$, given locally by

$$u^a \frac{\partial}{\partial x^a} - 2(G^a + \sigma u^a) \frac{\partial}{\partial u^a}.$$

The difference between two homogeneous sections is a homogeneous function on $T^\circ M$, so the corresponding sprays are projectively equivalent. The fundamental invariant of this equivalence class of sprays on $T^\circ M$ is just

$$\Pi^a_{bc} = \frac{\partial^2 G^a}{\partial u^b \partial u^c},$$

whence $G^a = \frac{1}{2}\Pi^a_{bc}u^b u^c$ by homogeneity. Furthermore,

$$G^0 = -\frac{1}{2(m-1)}\left(2\frac{\partial G^a}{\partial x^a} - G^a_b G^b_a\right) = -\frac{1}{2(m-1)}\mathfrak{R}_{cd}u^c u^d.$$

Suppose given a projective equivalence class of sprays on $T^\circ M$; then over each coordinate patch U on M there is a unique spray $\tilde{S}_U$ on $(T^\circ(\mathcal{V}M))|_U$ which satisfies the four conditions given earlier and generates the class by the construction just given. Since the conditions which determine $\tilde{S}_U$ are coordinate independent, and determine it uniquely, the $\tilde{S}_U$ agree on overlaps of coordinate patches, and therefore fit together to give a global spray. This is the *BTW*-spray of the projective equivalence class.

The Berwald connection coefficients of the *BTW*-spray of a projective equivalence class are

$$\tilde{\Gamma}^0_{0\alpha} = \tilde{\Gamma}^0_{\alpha 0} = 0, \quad \tilde{\Gamma}^0_{ab} = -\frac{1}{m-1}x^0\mathfrak{R}_{ab}, \quad \tilde{\Gamma}^a_{00} = 0, \quad \tilde{\Gamma}^a_{0b} = \tilde{\Gamma}^a_{b0} = \left(x^0\right)^{-1}\delta^a_b, \quad \tilde{\Gamma}^a_{bc} = \Pi^a_{bc};$$

these are of course the connection coefficients of the *BTW*-connection.

The *BTW*-spray of a reversible spray is itself reversible.

3.6. An application of the BTW-connection construction

To demonstrate the utility of the *BTW*-connection construction, we now show how to construct for isotropic sprays, essentially by algebraic means, functions which are constant along their paths; in generic cases the process will, at least in principle, generate sufficiently many such constants to integrate the sprays, that is, specify the paths completely. The idea is due to Grossman [13], but the implementation in terms of isotropic sprays is original.

First, an obvious remark: if T is a tensor which satisfies $\nabla_X T = 0$ for some covariant derivative operator ∇ then any scalar f that can be formed out of T is a first integral of the vector field X, that is, it satisfies $Xf = 0$. Here f need not depend linearly on T, and in the application cannot; a good example of what we have in mind is the traces of powers of a type (1, 1) tensor.

The appropriate general linear group acts on the tensor space by the corresponding tensor action. To say that f is a scalar is to say that it is invariant under this action. The dimension of a cross-section of the action, at least so far as 'generic' tensors are concerned, gives the number of independent scalars. The case of interest consists of tensors D^a_{bcd} which are trace-free and symmetric in the lower indices. Now $GL(m)$ acts irreducibly on this tensor space, which in this context means that there are no non-trivial scalars which depend linearly on the tensor argument. However, by a counting argument (which assumes that the isotropy group of the generic tensor is the identity) Grossman shows that there are many orbits and therefore generically many scalars.

The remainder of the problem is thus to find a tensor that satisfies $\nabla_S T = 0$ where S is a spray in the projective class. Suppose initially that S is R-flat. Now as we pointed out earlier, one of the Bianchi identities in spray geometry is $R^a_{bcd|e} = B^a_{bde|c} - B^a_{bce|d}$; when S is R-flat the left-hand side vanishes. Recall that $u^a_{|b} = 0$ and that $B^a_{bcd}u^c = 0$; then

$$\nabla_S B^a_{bcd} = u^e B^a_{bcd|e} = u^e B^a_{bed|c} = 0.$$

Thus a generic R-flat spray has enough first integrals to be explicitly solved, in principle.

We can turn this into a projective result by considering a projective class of isotropic sprays. It is easy enough to compute the curvatures of the normal *BTW*-connection: they can be expressed in terms of projective invariants of the corresponding projective class of sprays as follows. We write $\tilde{R}^\alpha_{\beta\gamma\delta}$ for the Riemann curvature and $\tilde{B}^\alpha_{\beta\gamma\delta}$ for the Berwald curvature; then

$$\tilde{R}^a_{bcd} = P^a_{bcd}, \quad \tilde{R}^0_{bcd} = \frac{1}{m-1}x^0\left(\mathfrak{R}_{bc|d} - \mathfrak{R}_{bd|c}\right),$$

$$\tilde{B}^a_{bcd} = D^a_{bcd}, \quad \tilde{B}^0_{bcd} = -\frac{1}{m-1} x^0 \mathfrak{R}_{bc,d},$$

all other components being zero in each case; the 'covariant derivatives' here are calculated with respect to the Π^a_{bc}. But when the sprays of the projective class are isotropic, and $m \geq 3$, $P^a_{bcd} = 0$; and it follows from this (as in the affine case) that $\mathfrak{R}_{bc|d} = \mathfrak{R}_{bd|c}$. Thus in the isotropic case the normal *BTW*-spray is R-flat, and so scalars formed from its Berwald tensor (which will include scalars formed from the Douglas tensor of the original projective class of sprays) will be first integrals of the *BTW*-spray. This gives a method of constructing first integrals of any spray in the projective class.

4. Cartan projective geometry

We now turn to Cartan's theory as it applies to the projective geometry of affine sprays. The Klein geometry on which the Cartan geometry is modelled is just projective space P^m, which is a homogeneous space of the projective group.

The projective group $\mathrm{PGL}(m+1)$ is the quotient of $\mathrm{GL}(m+1)$ by non-zero multiples of the identity. When m is even, $\mathrm{PGL}(m+1) \cong \mathrm{SL}(m+1)$; when m is odd, on the other hand, elements of $\mathrm{PGL}(m+1)$ may be identified with equivalence classes containing pairs of matrices $\pm g$ where $\det g = \pm 1$ according as the corresponding element of $\mathrm{PGL}(m+1)$ consists of matrices with positive or with negative determinant. We will take particular care in the discussion below to identify any differences between the two cases. We will in fact represent elements of $\mathrm{PGL}(m+1)$ by matrices g with $|\det g| = 1$, but we will bear it in mind that for m odd such a matrix is determined only up to sign.

Other authors, in describing Cartan projective geometries, take the underlying group to be $\mathrm{PSL}(m+1)$ instead: when m is even this is the same as $\mathrm{PGL}(m+1)$, but when m is odd the latter group is not connected, and $\mathrm{PSL}(m+1)$ is then its identity component. Using this subgroup when m is odd amounts to choosing an orientation for the model geometry (recall that P^m is orientable in this case); a corresponding Cartan geometry can then be constructed only when M is orientable. The use of $\mathrm{PGL}(m+1)$ avoids this restriction.

The Lie algebra of $\mathrm{PGL}(m+1)$ is $\mathfrak{sl}(m+1)$, as indeed is the Lie algebra of $\mathrm{PSL}(m+1)$.

The other group of importance in the definition of a Cartan projective geometry is the subgroup $\mathrm{H}_{m+1} \subset \mathrm{PGL}(m+1)$ which is the stabilizer of the point $[1, 0, \ldots, 0] \in \mathrm{P}^m$. In matrix representation its elements are matrices whose first column is zero below the diagonal. We denote by $\mathfrak{h}_{m+1}$ the Lie algebra of H_{m+1}.

4.1. Cartan projective connections

A Cartan projective geometry consists of a suitable principal H_{m+1}-bundle $P \to M$ and an $\mathfrak{sl}(m+1)$-valued 1-form ω on P, the connection form, satisfying the following conditions (see for example Sharpe ([23], Definition 5.3.1)):

1. the map $\omega_p : T_p P \to \mathfrak{sl}(m+1)$ is an isomorphism for each $p \in P$;
2. $R^*_h \omega = \mathrm{ad}(h^{-1})\omega$ for each $h \in \mathrm{H}_{m+1}$; and
3. $\langle A^\dagger, \omega \rangle = A$ for each $A \in \mathfrak{h}_{m+1}$, where $A^\dagger$ is the fundamental vector field corresponding to A.

Though this global, bundle definition of a Cartan connection is the most satisfying, in practice one usually works locally, in a gauge (as indeed Cartan himself did, in effect). By a gauge we simply mean a local section, say κ, of $P \to M$; the connection form in that gauge is $\kappa^*\omega$, a locally-defined $\mathfrak{sl}(m+1)$-valued 1-form on M. It is a consequence of the first condition above that a gauged connection form must have the property that for each point x in its domain the linear map $\rho \circ (\kappa^*\omega)(x) : T_x M \to \mathfrak{sl}(m+1)/\mathfrak{h}_{m+1}$, where $\rho : \mathfrak{sl}(m+1) \to \mathfrak{sl}(m+1)/\mathfrak{h}_{m+1}$ is the projection, is an isomorphism.

It follows from the conditions on ω itemized above that given two local gauges κ and $\hat{\kappa}$ with overlapping domains, the corresponding locally-defined matrices of forms $\kappa^*\omega$ and $\hat{\kappa}^*\omega$ on M are related by the transformation rule $\kappa^*\omega = \mathrm{ad}(h^{-1})(\hat{\kappa}^*\omega) + h^*(\theta_{\mathrm{H}_{m+1}})$, where $\theta_{\mathrm{H}_{m+1}}$ is the Maurer–Cartan form on H_{m+1} and h is the local H_{m+1}-valued function relating the two gauges κ and $\hat{\kappa}$, such that $\kappa(x) = R_{h(x)}\hat{\kappa}(x)$. If the domain of h is simply connected we can consistently choose a matrix-valued function to represent it, in which case the transformation rule may be written as

$$\kappa^*\omega = h^{-1}(\hat{\kappa}^*\omega)h + h^{-1}\mathrm{d}h;$$

since h enters this equation quadratically, the possible sign indeterminacy in its matrix representation has no effect.

Conversely, given a covering of M by local gauges and local matrices of forms satisfying this transformation rule, it is possible to reconstruct the principal bundle in terms of transition functions, as we will explain more fully below.

One advantage of working in a gauge is that it may be possible to select a particularly simple gauged connection form, and this is certainly the case for a projective connection. We assume the existence of a Cartan connection, and start with an arbitrary gauged connection form $\hat{\kappa}^*\omega$ which without loss of generality we take to be defined in a coordinate patch. We can write $\hat{\kappa}^*\omega$ as a matrix-valued form as follows:

$$\hat{\kappa}^*\omega = \begin{pmatrix} \hat{\omega}_0^0 & \hat{\omega}_b^0 \\ \hat{\omega}_0^a & \hat{\omega}_b^a \end{pmatrix};$$

each entry in the matrix is a locally defined 1-form on M. Now we can identify $\mathfrak{sl}(m+1)/\mathfrak{h}_{m+1}$ with $\mathfrak{R}^m$, so that the linear map $\rho \circ (\kappa^*\omega)(x) : T_x M \to \mathfrak{sl}(m+1)/\mathfrak{h}_{m+1} = \mathbf{R}^m$ is just given by the vector-valued 1-form $(\hat{\omega}_0^a)$. It is a consequence of the defining conditions for a connection form that this map is an isomorphism for each x, or in other words if we set $\hat{\omega}_0^a = \hat{\omega}_{0b}^a \mathrm{d}x^b$ then the $m \times m$ matrix $(\hat{\omega}_{0b}^a)$ is nonsingular. We will show that by a change of gauge we can transform $\hat{\omega}_0^a$ to $\mathrm{d}x^a$. To see this, note first that if h is a matrix of the form

$$h = \begin{pmatrix} h_0^0 & h_b^0 \\ 0 & h_b^a \end{pmatrix},$$

then its inverse is given by

$$h^{-1} = \begin{pmatrix} \bar{h}_0^0 & -\bar{h}_0^0 h_c^0 \bar{h}_b^c \\ 0 & \bar{h}_b^a \end{pmatrix}$$

where the overbar signifies (an element of) the inverse matrix ($m \times m$ or 1×1 as the case may be). Note that $\det h = h_0^0 \det(h_b^a)$. We denote the matrix elements of $\kappa^*\omega = h^{-1}(\hat{\kappa}^*\omega)h + h^{-1}\mathrm{d}h$ by ω_β^α, so that $\omega_\beta^\alpha = \bar{h}_\gamma^\alpha \hat{\omega}_\delta^\gamma h_\beta^\delta + \bar{h}_\gamma^\alpha \mathrm{d}h_\beta^\gamma$; then $\omega_0^a = h_0^0 \bar{h}_b^a \hat{\omega}_0^b$. In order to make $\omega_0^a = \mathrm{d}x^a$ we must therefore solve the equations $h_0^0 \bar{h}_c^a \hat{\omega}_{0b}^c = \delta_b^a$ for elements h_0^0, h_b^a of a matrix h representing an element of H_{m+1}. From these equations we obtain, by taking determinants, $(h_0^0)^{m+1}(\det h)^{-1} \det \hat{\omega}_0 = 1$, where $\hat{\omega}_0 = (\hat{\omega}_{0b}^a)$ and $\det \hat{\omega}_0 \neq 0$. If m is even we require that $\det h = 1$, so $h_0^0 = (\det \hat{\omega}_0)^{-1/(m+1)}$; this solution is unique. On the other hand, if m is odd we require only that $|\det h| = 1$; then a necessary condition for a solution to exist is that $\det h$ and $\det \hat{\omega}_0$ have the same sign: if $\det \hat{\omega}_0 > 0$ then we must take $\det h = 1$ and so $h_0^0 = \pm(\det \hat{\omega}_0)^{-1/(m+1)}$, whereas if $\det \hat{\omega}_0 < 0$ then we must take $\det h = -1$ and so $h_0^0 = \pm(\det(-\hat{\omega}_0))^{-1/(m+1)}$. In either case, we obtain a unique solution for h_0^0 up to sign. If we set $h_b^a = h_0^0 \hat{\omega}_{0b}^a$ we obtain (for any choice of h_a^0) an element of H_{m+1} such that $\omega_0^a = \mathrm{d}x^a$. We can combine the solutions for even and odd m in one formula by setting

$$h_0^0 = \frac{\det \hat{\omega}_0}{|\det \hat{\omega}_0|} |\det \hat{\omega}_0|^{-1/(m+1)};$$

it must be understood that when m is odd both $(m+1)$-th roots must be taken.

There is still some freedom in the choice of gauge, which we can eliminate as follows. The gauge transformation rule gives $\omega_0^0 = \hat{\omega}_0^0 - h_b^0 \bar{h}_a^b \hat{\omega}_0^a + \bar{h}_0^0 \mathrm{d}h_0^0$; so if we define h_a^0 by $h_a^0 \mathrm{d}x^a = h_0^0 \hat{\omega}_0^0 + \mathrm{d}h_0^0$, we will have $\omega_0^0 = 0$. Therefore, for any projective connection on a manifold M there is a covering of M by coordinate patches and for each patch a unique choice of gauge with respect to which the gauged connection form is

$$\begin{pmatrix} 0 & \omega_b^0 \\ \mathrm{d}x^a & \omega_b^a \end{pmatrix},$$

where $\omega_a^a = 0$. We call such a gauge the standard gauge for those coordinates.

We can use the standard gauges to find transition functions for the bundle $P \to M$, and thus define it implicitly. Let (ω_β^α), $(\hat{\omega}_\beta^\alpha)$ be gauged connection forms for a projective connection, in standard gauge with respect to two overlapping coordinate patches with coordinates (x^a) and $(\hat{x}^a)$. By considering the gauge transformation of $(\hat{\omega}_\beta^\alpha)$ to standard form with respect to the coordinates (x^a) we have $\omega^\alpha_\beta = \bar{h}^\alpha_\gamma \hat{\omega}^\gamma_\delta h^\delta_\beta + \bar{h}^\alpha_\gamma \mathrm{d}h^\gamma_\beta$ with

$$h^0_0 = \varepsilon_J |J|^{-1/(m+1)}, \quad h^a_b = \varepsilon_J |J|^{-1/(m+1)} J^a_b, \quad h^0_c \mathrm{d}x^c = \varepsilon_J d|J|^{-1/(m+1)}$$

where as before (J^a_b) is the Jacobian matrix of the coordinate transformation, J is the Jacobian determinant, and $\varepsilon_J = J/|J|$. That is,

$$h = \varepsilon_J |J|^{-1/(m+1)} \begin{pmatrix} 1 & -\dfrac{1}{m+1}\dfrac{\partial \log |J|}{\partial x^b} \\ 0 & J^a_b \end{pmatrix}.$$

If m is even then this is a matrix in $\mathrm{SL}(m+1)$, but if m is odd then both $(m+1)$-th roots must be taken and the result is a pair of matrices in $\mathrm{GL}(m+1)$ whose determinants are of absolute value 1. In either case we obtain an element of $\mathrm{H}_{m+1} \subset \mathrm{PGL}(m+1)$.

Thus given a manifold M with a Cartan projective connection we have an open covering of M by coordinate neighbourhoods $\{U_\lambda\}$ and smooth maps $h_{\mu\lambda} : U_\lambda \cap U_\mu \to \mathrm{H}_{m+1}$ determined by the gauge transformation between the gauged connections in standard form on the two coordinate patches. The maps $h_{\mu\lambda}$ satisfy

$$h_{\nu\mu} h_{\mu\lambda} = h_{\nu\lambda} \quad \text{on } U_\lambda \cap U_\mu \cap U_\nu;$$

this follows from their construction, but can also be established easily from the explicit formula. They are therefore transition functions in the definition of a principal H_{m+1}-bundle P; then the connection form in standard gauge will be the pull-back by a suitable local section of a global Cartan connection form on P, and we regain the principal bundle definition of the projective connection.

The transition functions are derived from consideration of the left column of the gauged connection form alone. Using the transition functions and assuming that we have a globally defined Cartan connection form we can compute the coordinate transformation properties of the remaining entries in the gauged connection form. We find, in particular, that if we set $\omega^c_a = \omega^c_{ab} \mathrm{d}x^b$ then

$$\omega^c_{ab} = \bar{J}^d_a \bar{J}^e_b (J^c_f \hat{\omega}^f_{de} - J^c_{de}) + \frac{1}{m+1} \frac{\partial \log |J|}{\partial x^d} (\bar{J}^d_a \delta^c_b + \bar{J}^d_b \delta^c_a),$$

that is, that the symmetric part of ω^c_{ab}, $\omega^c_{(ab)} = \frac{1}{2}\left(\omega^c_{ab} + \omega^c_{ba}\right)$, transforms as the fundamental descriptive invariant of a projective equivalence class. However, although $\omega^a_{ab} = 0$, it is not necessarily the case that $\omega^a_{(ab)} = 0$.

4.2. Geodesics

The definition of a geodesic in a Cartan geometry depends on the notion of the development of a curve in M into a curve in the Klein geometry G/H on which the Cartan geometry is modelled. Let ω be the Cartan connection form, and κ a gauge. A curve $x(t)$ in M defines a curve X_κ in $\mathfrak{g}$ by

$$X_\kappa(t) = \langle \dot{x}(t), \kappa^* \omega \rangle.$$

We assume that G is a matrix group, for simplicity. Let $g(t)$ be a curve in G which is a solution of the matrix differential equation $\dot{g} = g X_\kappa$, and set $\xi(t) = g(t)\xi_0$ where ξ_0 is the point in the homogeneous space of which H is the stabilizer. It is easy to see that, unlike $g(t)$, $\xi(t)$ is unchanged by a change of gauge: in fact $g(t)$ changes to $g(t)h(t)$ where $h(t)$ is a curve in H. Then $\xi(t)$ is a development of $x(t)$. It is clear that there is a development of a given curve in M through each point of G/H.

We define a geodesic of a Cartan projective connection as a curve in M whose development in projective space P^m is a straight line. We will now find the geodesics of a Cartan projective connection. We take a connection form in standard gauge and write $X(t)$ for

$$\begin{pmatrix} 0 & \omega^0_{bc}\dot{x}^c \\ \dot{x}^a & \omega^a_{bc}\dot{x}^c \end{pmatrix}.$$

Then any development $\xi(t)$ of $x(t)$ into P^m is given by $\xi(t) = g(t)\xi_0$ where $\xi_0 = [1, 0, \ldots, 0]$ and $g(t)$ satisfies $\dot{g} = gX$. Now $\xi(t)$ is a curve in projective space P^m; if we wish to consider the equation defining it as a vector equation we must introduce an arbitrary non-vanishing scalar factor, say $\phi(t)$. That is, the development of $x(t)$ is $[u(t)]$ where $u(t)$ is a curve in $\mathbf{R}^{m+1}$ such that $u(t) = \phi(t)g(t)e_0$ where $e_0 = (1, 0, \ldots, 0)$. Let us assume that the parametrization is chosen such that the straight line in P^m is given by $\ddot{u} = 0$. Then

$$0 = \frac{\mathrm{d}^2}{\mathrm{d}t^2}(\phi g)e_0 = (\ddot{\phi}g + 2\dot{\phi}\dot{g} + \phi\ddot{g})e_0 = g(\ddot{\phi}I + 2\dot{\phi}X + \phi(\dot{X} + X^2))e_0.$$

Thus $x(t)$ will be a geodesic if and only if there is some non-vanishing function $\phi(t)$ such that

$$(\ddot{\phi}I + 2\dot{\phi}X + \phi(\dot{X} + X^2))e_0 = 0.$$

This is equivalent to a pair of equations, one vector and one scalar:

$$\ddot{x}^c + \omega^c_{ab}\dot{x}^a\dot{x}^b = -2(\dot{\phi}/\phi)\dot{x}^c, \quad \ddot{\phi} + \phi\,\omega^0_{ab}\dot{x}^a\dot{x}^b = 0.$$

From the first of these we see that a global Cartan projective connection determines a restricted path space whose paths are its geodesics; and conversely, given a restricted path space there is a global Cartan connection (in fact there are many) whose geodesics are its paths. In fact these geodesic equations are exactly the same as the equations for the geodesics of a *TW*-connection obtained earlier, with the substitutions of $\omega^c_{(ab)}$ for $\tilde{\Gamma}^c_{ab}$, $\omega^0_{(ab)}$ for α_{ab}, and ϕ for x^0.

4.3. Normalizing the Cartan projective connection

The curvature of a Cartan projective connection is the $\mathfrak{sl}(m+1)$-valued 2-form (Ω^α_β) where

$$\Omega^\alpha_\beta = \mathrm{d}\omega^\alpha_\beta + \omega^\alpha_\gamma \wedge \omega^\gamma_\beta.$$

The vanishing of the curvature is the necessary and sufficient condition for the Cartan geometry to be locally diffeomorphic to P^m, the Klein geometry on which it is modelled.

The torsion of the Cartan connection is the $\mathbf{R}^m$-valued 2-form (Ω^a_0).

We can consider curvature and torsion in a local gauge; the definitions are formally the same. Under a change of gauge the curvature transforms by $\Omega^\alpha_\beta = \bar{h}^\alpha_\gamma \hat{\Omega}^\gamma_\delta h^\delta_\beta$; it follows that the torsion transforms by $\Omega^a_0 = h^0_0 \bar{h}^a_b \hat{\Omega}^b_0$. Of particular interest are connections with vanishing torsion. It is clear from the transformation rule that this is a gauge-independent property of a connection. If we take a connection in standard gauge, its torsion is just

$$\Omega^a_0 = -\omega^a_{bc}\mathrm{d}x^b \wedge \mathrm{d}x^c;$$

so the connection has zero torsion if and only if ω^a_{bc} is symmetric in its lower indices. Thus we can ensure that a Cartan projective connection associated with a given restricted path space is torsion-free by taking $\omega^c_{ab} = \Pi^c_{ab}$.

One of the achievements of Cartan [3] was to show that although many projective connections give rise to the same restricted path space, there is a distinguished torsion-free connection which can be specified uniquely by conditions on its curvature.

Assume that we are given a restricted path space, and a torsion-free Cartan projective connection adapted to it as just described. We will show how to determine the remaining elements of the Cartan connection by further conditions on the curvature, so as to fix them uniquely. These conditions will be specified in terms of the standard gauge, but will be gauge-independent, which is to say that if they hold in one gauge they hold in any; we can then be sure that a connection which satisfies the conditions and is uniquely determined by them will be globally defined.

By assumption, in standard gauge the gauged connection and curvature forms are given by

$$\begin{pmatrix} 0 & \omega^0_b \\ \mathrm{d}x^a & \Pi^a_{bc}\mathrm{d}x^c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \Omega^0_0 & \Omega^0_b \\ 0 & \Omega^a_b \end{pmatrix}.$$

First,

$$\Omega^0_0 = -\omega^0_{bc}\mathrm{d}x^b \wedge \mathrm{d}x^c$$

where of course $\omega^0_b = \omega^0_{bc}\mathrm{d}x^c$; thus if we take ω^0_{bc} to be symmetric we will have $\Omega^0_0 = 0$. Note that if the connection is torsion-free then Ω^0_0 is unchanged by a gauge transformation, so this property is gauge-independent for torsion-free connections. Then

$$\Omega^a_b = \frac{1}{2}\left(\mathfrak{R}^a_{bcd} + \delta^a_c\omega^0_{bd} - \delta^a_b\omega^0_{cd}\right)\mathrm{d}x^c \wedge \mathrm{d}x^d$$

where $\mathfrak{R}^a_{bcd}$ is the curvature 'tensor' derived from the Π^a_{bc}. Thus if $\Omega^a_b = \frac{1}{2}\Omega^a_{bcd}\mathrm{d}x^c \wedge \mathrm{d}x^d$, with Ω^a_{bcd} skew in c and d,

$$\Omega^a_{bcd} = \mathfrak{R}^a_{bcd} + \delta^a_c\omega^0_{bd} - \delta^a_d\omega^0_{bc}.$$

We can make Ω^a_{bcd} trace-free ($\Omega^c_{bcd} = 0$) by choosing $(m-1)\omega^0_{bc} = -\mathfrak{R}_{bc}$, in which case

$$\Omega^a_{bcd} = \mathfrak{R}^a_{bcd} - \frac{1}{m-1}\left(\mathfrak{R}_{bd}\delta^a_c - \mathfrak{R}_{bc}\delta^a_d\right) = P^a_{bcd},$$

the projective curvature tensor. The condition that this tensor be trace-free is gauge-independent.

The conditions that $\Omega^0_0 = 0$ and $\Omega^c_{bcd} = 0$ determine ω uniquely. That is to say, given a restricted path space, there is a unique globally defined torsion-free $\mathfrak{sl}(m+1)$-valued Cartan projective connection form with the paths as its geodesics, whose curvature satisfies $\Omega^0_0 = 0$ and $\Omega^c_{bcd} = 0$. It is called the normal projective connection form, and in the standard gauge it is given by

$$\omega = \begin{pmatrix} 0 & -\dfrac{1}{m-1}\mathfrak{R}_{bc}\mathrm{d}x^c \\ \mathrm{d}x^a & \Pi^a_{bc}\mathrm{d}x^c \end{pmatrix}.$$

It is easy to verify that

$$\Omega^0_b = \frac{1}{m-1}\mathfrak{R}_{b[c|d]}\mathrm{d}x^c \wedge \mathrm{d}x^d,$$

where the brackets in the suffix indicate skew-symmetrization and the solidus 'covariant differentiation' with respect to the fundamental invariant. The curvature of the normal projective connection is therefore

$$\Omega = \begin{pmatrix} 0 & \dfrac{1}{m-1}\mathfrak{R}_{b[c|d]}\mathrm{d}x^c \wedge \mathrm{d}x^d \\ 0 & \dfrac{1}{2}P^a_{bcd}\mathrm{d}x^c \wedge \mathrm{d}x^d \end{pmatrix}.$$

5. The Cartan bundle

In this section we will describe a canonical procedure, starting with a manifold M, for constructing a principal bundle $\mathcal{C}M \to M$ with structure group $\mathrm{H}_{m+1} \subset \mathrm{PGL}(m+1)$ where $m = \dim M$. This procedure does not require a connection (of any kind) for the construction of the principal bundle: it just uses geometric properties of the manifold M. Nevertheless, the bundle constructed in this way has the same transition functions as one built synthetically using the transformation properties of a Cartan projective connection obtained above. We emphasise that the construction works whether or not M is orientable, and whether m is even or odd.

In order to construct a principal bundle as the domain for a Cartan projective connection, we will first consider the problem from Cartan's point of view: that at each point of M there should be attached a projective space of the same dimension m. Of course there is already a projective space of dimension $m-1$, namely the fibre of the projective tangent bundle $\mathrm{P}TM$, but this is too small for our purposes. There are, however, projective spaces of dimension m attached to each point of the volume bundle $\mathcal{V}M$, and so we will describe a mechanism for transferring these consistently to M. This mechanism initially works on the underlying vector spaces, and so creates a vector bundle over M whose fibre dimension is $m+1$; this is just the bundle $\mathcal{W}M \to M$ which we introduced earlier, which is the quotient of the tangent bundle to the volume bundle $\tau_{\mathcal{V}M} : T(\mathcal{V}M) \to \mathcal{V}M$ under the derivative of the action μ_s. It is called the Cartan algebroid, for reasons which will be explained shortly.

5.1. The Cartan algebroid

We start with the tangent bundle to the volume bundle, $\tau_{\mathcal{V}M} : T(\mathcal{V}M) \to \mathcal{V}M$. Let $\mu_{s*} : T(\mathcal{V}M) \to T(\mathcal{V}M)$ be the derivative of the action μ_s on the fibres of $\nu : \mathcal{V}M \to M$, and let $\mathcal{W}M$ be the space of orbits of μ_*; then $\mathcal{W}M$ is a manifold with coordinates (x^a, u^a, w) where $w = (x^0)^{-1}u^0$, that is, if $\xi \in T(\mathcal{V}M)$ and $[\xi]$ is its μ_*-orbit,

$$w([\xi]) = \frac{u^0(\xi)}{x^0(\xi)}$$

(recall that $x^0 > 0$). It is clear that the action μ_* respects the fibration $\nu_* : T(\mathcal{V}M) \to TM$, so that $\mathcal{W}M$ is fibred over M. If ρ, τ are the two projections from $\mathcal{W}M$ to TM and M respectively, and if χ satisfies $\nu_* = \rho \circ \chi$, then we have the following diagram.

$$\begin{tikzcd}
T(\mathcal{V}M) \arrow[r, "\chi"] \arrow[d, "\tau_{\mathcal{V}M}"'] & \mathcal{W}M \arrow[r, "\rho"] \arrow[d, "\tau"'] & TM \arrow[d, "\tau_M"'] \\
\mathcal{V}M \arrow[r, "\nu"'] & M \arrow[r, "="'] & M
\end{tikzcd}$$

Furthermore, the action μ_{s*} is linear on the fibres of $\tau_{\mathcal{V}M}$, so $\tau : \mathcal{W}M \to M$ is a vector bundle, and the projection $\chi : T(\mathcal{V}M) \to \mathcal{W}M$ is linear on the fibres. In fact χ is a fibrewise isomorphism, as is evident from the coordinate representation

$$u^a \circ \chi = u^a, \quad w \circ \chi = (x^0)^{-1}u^0$$

of the isomorphism $T_{[\pm\theta]}(\mathcal{V}M) \to \mathcal{W}_{\nu[\pm\theta]}M$. Two other significant facts about $\mathcal{W}M$ are worth mentioning. First, the fibres of $\chi : T(\mathcal{V}M) \to \mathcal{W}M$ are the integral curves of Υ^{C}, the complete lift of Υ to $T(\mathcal{V}M)$. Second, we may identify $T(\mathcal{V}M)$ with the pullback $\nu^*(\mathcal{W}M)$ by the map $\xi \mapsto (\tau_{\mathcal{V}M}(\xi), \chi(\xi))$.

We will denote the vector space of vector fields on $\mathcal{V}M$ by $\mathfrak{X}(\mathcal{V}M)$, and the subspace of vector fields projectable to sections of $\tau : \mathcal{W}M \to M$ by $\mathfrak{X}_M(\mathcal{V}M)$. The latter is a proper subspace of the space of 'projectable vector fields' in the ordinary sense, that is those projectable to vector fields on M: for instance ∂_0 projects to a vector field on M (the zero field) but does not project to a section of τ. In fact $\mathfrak{X}_M(\mathcal{V}M)$, although not a module over the ring of all functions on $\mathcal{V}M$, is a module over the sub-ring of functions constant on the fibres of ν. If $X \in \mathfrak{X}_M(\mathcal{V}M)$ then X must satisfy $X_{\mu_s[\pm\theta]} = \mu_{s*}(X_{[\pm\theta]})$, and a local basis for the module is given by $\{\Upsilon, \partial/\partial x^a\}$. The global condition for $X \in \mathfrak{X}_M(\mathcal{V}M)$ is $[X, \Upsilon] = 0$, and the Jacobi identity then implies that $\mathfrak{X}_M(\mathcal{V}M)$ is a Lie subalgebra of $\mathfrak{X}(\mathcal{V}M)$. We will denote the image sections of the local basis by $\{e_0, e_a\}$ (where of course e_0, as the image of Υ, is defined globally).

It follows from the preceding remarks that the bundle $\tau : \mathcal{W}M \to M$ is a Lie algebroid, with base dimension m and fibre dimension $m + 1$. If $\underline{\chi} : \mathfrak{X}_M(\mathcal{V}M) \to \mathrm{sect}(\tau)$ denotes the induced map of sections (so that $\underline{\chi}(X)_{\nu[\pm\theta]} = \chi(X_{[\pm\theta]})$) then $\underline{\chi}$ is a module isomorphism, and so may be used to define a Lie bracket on sections of τ; the map $\rho : \mathcal{W}M \to TM$ is the anchor map. In fact $\tau : \mathcal{W}M \to M$ is the Atiyah algebroid of $\nu : \mathcal{V}M \to M$, considered as a principal bundle. This explains why we call this bundle the Cartan algebroid of M. However, it contains no more information than the canonical tangent bundle algebroid because the global section e_0 is in its centre: $[e_0, e] = 0$ for any section e.

The quotient of the Cartan algebroid by the equivalence relation of non-zero multiplication in the fibres is the Cartan projective bundle $\mathrm{P}\mathcal{W}M$; this is the projective bundle with m-dimensional fibres that we need.

We have proposed that the construction of the Cartan projective bundle $\mathrm{P}\mathcal{W}M$ corresponds to Cartan's notion of attaching a projective space to each point of the manifold M. To make this correspondence even clearer, we now point out just how firmly the projective spaces are attached to M by our construction: the Cartan projective bundle is actually soldered to M. (If one is given a Cartan projective connection one can use it to define a soldering: but here we are dealing just with the Cartan projective bundle and make no appeal to the existence of a connection.)

The following definition is taken from Kobayashi [16]. A fibre bundle $B \to M$ with standard fibre F is soldered to M if the following conditions are satisfied:

- $\dim F = \dim M$;
- B admits a cross-section which will be identified with M;
- let $\tilde{T}M$ be the space of all tangent vectors to F_x (fibre over $x \in M$) for all $x \in M$: then TM is isomorphic to $\tilde{T}M$; more precisely, there is a mapping σ of TM onto $\tilde{T}M$ such that, for each x in M, σ is a non-singular linear mapping of T_xM onto the space of all tangent vectors to F_x at x.

We now show that $\mathrm{P}\mathcal{W}M$ is soldered to M according to this definition.

The condition on the dimensions is clearly satisfied. We know that $\mathrm{P}\mathcal{W}M \to M$ admits a global section, namely $[e_0]$. Notice that the projection $\rho : \mathcal{W}M \to TM$ maps the section e_0 of $\mathcal{W}M$ to the zero section of TM, and more generally that the kernel of ρ (as a vector bundle over M) is just the 1-dimensional sub-bundle of $\mathcal{W}M$ spanned by e_0. Let $V_0(\mathcal{W}M)$ be the restriction to the section e_0 of the vertical sub-bundle of $T(\mathcal{W}M)$, and $V_0(TM)$ the restriction to the zero section of the vertical sub-bundle of TTM, which can of course be canonically identified with TM. Then ρ_* restricts to a linear map of $V_0(\mathcal{W}M)$ onto $V_0(TM)$, which is just ρ in a different guise; its kernel is again spanned by e_0, considered now as a section of $V_0(\mathcal{W}M)$ via its vertical lift e_0^{V}. We will show that $V_0(\mathrm{P}\mathcal{W}M)$, the restriction to the section $[e_0]$ of the vertical sub-bundle of $T(\mathrm{P}\mathcal{W}M)$, is canonically isomorphic to $V_0(\mathcal{W}M)/\langle e_0^{\mathrm{V}}\rangle$, the quotient bundle of $V_0(\mathcal{W}M)$ by the 1-dimensional sub-bundle spanned by e_0^{V}. It will follow that $V_0(\mathrm{P}\mathcal{W}M)$ is canonically isomorphic to $V_0(TM)$, and therefore to TM, as required.

This is simply a matter of identifying the tangent space to a projective space in an appropriate way. Let W be a vector space with distinguished non-zero element e, $\mathrm{P}W$ the corresponding projective space with distinguished point $[e]$, and $\pi : W \to \mathrm{P}W$ the projection. Then $\pi_* : T_eW \to T_{[e]}(\mathrm{P}W)$ is a surjective linear map whose kernel is the 1-dimensional subspace of T_eW which is the tangent space to the ray through e; and T_eW is canonically isomorphic to W, with the tangent space to the ray through e corresponding to the 1-dimensional subspace of W spanned by e itself. Thus $T_{[e]}(\mathrm{P}W)$ is isomorphic to the quotient space $W/\langle e\rangle$, and the result follows.

It is worth noticing that the soldering isomorphism is canonical only because there is a canonical way of choosing a representative of the projective point $[e_0]$, that is, because $\mathcal{W}M$ has a canonical global section e_0.

5.2. The Cartan principal bundles

By a frame of a vector bundle we mean an ordered basis of a fibre. We define an equivalence relation on frames of the Cartan algebroid as follows. Let (ζ_α) and $(\bar{\zeta}_\alpha)$ be frames of $\mathcal{W}M$ at some point $x \in M$; we let $(\bar{\zeta}_\alpha) \equiv (\zeta_\alpha)$ if there is a non-zero real number λ such that $\bar{\zeta}_\alpha = \lambda\zeta_\alpha$. The corresponding equivalence class will be denoted by $[\zeta_\alpha]$, and is a reference $(m+1)$-simplex for the m-dimensional projective space $\mathrm{P}\mathcal{W}_xM$. The bundle containing all these equivalence classes at all points of M will be denoted by $\mathcal{S}_{\mathcal{W}}M$: it is a principal $\mathrm{PGL}(m+1)$-bundle over M. If the first element ζ_0 of such an equivalence class is a multiple of the global vector section e_0 then we will call it a Cartan simplex. We will let $\mathcal{C}M \subset \mathcal{S}_{\mathcal{W}}M$ be the bundle containing all the Cartan simplices, and call it the Cartan bundle: it is a principal H_{m+1}-bundle over M, and is a reduction of $\mathcal{S}_{\mathcal{W}}M$.

As Cartan says: 'It is natural to take each point of the manifold to be one of the vertices of the frame attached at that point'; this corresponds precisely to restricting one's attention to Cartan simplices, having first identified M with the global section $[e_0]$ of $\mathrm{P}\mathcal{W}M$ as specified in the definition of soldering.

The Cartan projective bundle $\mathrm{P}\mathcal{W}M$ is an associated bundle of the principal bundle $\mathcal{C}M$, using the representation of H_{m+1} as a group of automorphisms of the standard fibre P^m.

We will now calculate the transition functions for the Cartan bundle $\mathcal{C}M$ relative to local trivializations of the form $[e_\alpha]$, where (e_α) is a local frame field for the Cartan algebroid $\mathcal{W}M$ which is the image of the local frame field $(\Upsilon, \partial/\partial x^a)$ on $\mathcal{V}M$, which in turn is an ordered local basis of the module $\mathfrak{X}_M(\mathcal{V}M)$ of vector fields projectable to $\mathcal{W}M$. In fact if (e_α), $(\hat{e}_\alpha)$ are two such local frame fields for $\mathcal{W}M$, corresponding to coordinates (x^a), $(\hat{x}^a)$ on overlapping coordinate patches U, $\hat{U}$ on M, and we define a $\mathrm{GL}(m+1)$-valued function G on $U \cap \hat{U}$ by $e_\alpha = G^\beta_\alpha\hat{e}_\beta$, then the transition function for $U \cap \hat{U}$ is just $[G]$, the projection of G into $\mathrm{PGL}(m+1)$.

We must first find the transformation law for the above local basis of $\mathfrak{X}_M(\mathcal{V}M)$ with respect to coordinate transformations on M. Suppose that (U, x^a) and $(\hat{U}, \hat{x}^a)$ are overlapping coordinate patches on M, and that $(\nu^{-1}(U), x^\alpha)$ and $(\nu^{-1}(\hat{U}), \hat{x}^\alpha)$ are the corresponding coordinate patches on $\mathcal{V}M$. Then from $\hat{x}^0 = |J|^{-1/(m+1)}x^0$ we obtain

$$\begin{aligned}\frac{\partial}{\partial x^a} &= J_a^b\frac{\partial}{\partial \hat{x}^b} + \frac{\partial \hat{x}^0}{\partial x^a}\frac{\partial}{\partial \hat{x}^0} = J_a^b\frac{\partial}{\partial \hat{x}^b} + \frac{1}{\hat{x}^0}\frac{\partial \hat{x}^0}{\partial x^a}\Upsilon\\ &= J_a^b\frac{\partial}{\partial \hat{x}^b} - \frac{1}{m+1}\frac{\partial \log|J|}{\partial x^a}\Upsilon.\end{aligned}$$

As we have noted before, Υ is a global vector field and is unchanged by the coordinate transformation. To obtain the corresponding transformation for the e_α we have merely to replace $\partial/\partial x^a$ by e_a and Υ by e_0 in these formulae. Thus

$$G = \begin{pmatrix} 1 & -\dfrac{1}{m+1}\dfrac{\partial \log|J|}{\partial x^b}\\ 0 & J_b^a\end{pmatrix}.$$

So in fact G takes its values in the affine group $\mathrm{A}(m)$ (in its standard representation in $\mathrm{GL}(m+1)$).

The transition function for $\mathcal{C}M$ with respect to local trivializations $[e_\alpha]$, $[\hat{e}_\alpha]$, is just the projective equivalence class of G. We may represent this projective class by a single matrix with determinant 1 for m even, or by a pair of matrices with determinant ±1 for m odd, as before. Note that $\det G = J$. When m is even we can form $(\det G)^{-1/(m+1)} = J^{-1/(m+1)}$ whatever the sign of $\det G$, and then $(\det G)^{-1/(m+1)}G$ is the unique member of the projective equivalence class of G whose determinant is 1. When m is odd, on the other hand, we must treat the cases $\det G > 0$ and $\det G < 0$ differently. In the first case we can form $(\det G)^{-1/(m+1)} = J^{-1/(m+1)}$, and then $(\det G)^{-1/(m+1)}G$ gives the two members of the projective equivalence class of G with determinant 1. In the second case we can form $(-\det G)^{-1/(m+1)} = (-J)^{-1/(m+1)}$, and then $(-\det G)^{-1/(m+1)}G$ gives the two members of the projective equivalence class of G with determinant -1. These prescriptions can be combined in the single formula

$$[G] \equiv \varepsilon_J|J|^{-1/(m+1)}\begin{pmatrix} 1 & -\dfrac{1}{m+1}\dfrac{\partial \log|J|}{\partial x^b}\\ 0 & J_b^a\end{pmatrix},$$

as before, which shows that the transition functions for $\mathcal{C}M$ corresponding to the given local trivializations take their values in H_{m+1} and are exactly the functions obtained from the consideration of Cartan projective connections in standard gauge in the previous section. We conclude that the principal H_{m+1}-bundle implicitly defined via Cartan projective connections is (up to equivalence) $\mathcal{C}M$.

5.3. Further structure of the Cartan bundle

We have just defined the Cartan bundle $\mathcal{C}M$ as a principal H_{m+1}-bundle over M, where H_{m+1} is the projective image of the subgroup of $\mathrm{GL}(m+1)$ consisting of matrices with zeros below the diagonal in the first column. We will now show that the Cartan bundle has a second principal bundle structure; this new structure will be important when we come to generalizing the Cartan connection to arbitrary sprays.

In order to motivate the construction we go back to consider the relevant Klein geometry. In the affine case this is m-dimensional real projective space P^m, represented as a homogeneous space of the projective group $\mathrm{PGL}(m+1)$, with stabilizer subgroup H_{m+1}.

We now turn our attention to $\mathrm{P}T(\mathrm{P}^m)$, the projective tangent bundle of P^m, and show that it too is a homogeneous space of $\mathrm{PGL}(m+1)$. Each point of $\mathrm{P}T(\mathrm{P}^m)$ consists of a line through the origin in $\mathbf{R}^{m+1}$ and a 2-plane containing the line. The group $\mathrm{PGL}(m+1)$ acts transitively on $\mathrm{P}T(\mathrm{P}^m)$. The stabilizer of the point consisting of the first coordinate axis and the 2-plane containing the first two coordinate axes is the subgroup K_{m+1} of $\mathrm{PGL}(m+1)$ which is the image of the subgroup of $\mathrm{GL}(m+1)$ consisting of matrices with zeros below the main diagonal in the first and second columns; so we can identify $\mathrm{P}T(\mathrm{P}^m)$ with $\mathrm{PGL}(m+1)/\mathrm{K}_{m+1}$. It is hardly necessary to point out that K_{m+1} is a subgroup of H_{m+1}; we can identify the coset space $\mathrm{H}_{m+1}/\mathrm{K}_{m+1}$ with P^{m-1}, the standard fibre of $\mathrm{P}T(\mathrm{P}^m) \to \mathrm{P}^m$.

We now show that $\mathcal{C}M$ is a principal K_{m+1}-bundle over $\mathrm{P}TM$.

By definition $\mathcal{C}M$ consists of the Cartan simplices of the Cartan algebroid $\mathcal{W}M$, that is, the simplices with first element a multiple of the global vector section e_0 of $\tau : \mathcal{W}M \to M$. Let $[\zeta_\alpha]$ be a Cartan simplex of $\mathcal{W}M$ at some point $x \in M$, so that ζ_0 is a multiple of $(e_0)_x$. Now ζ_1 is an element of $\mathcal{W}_x M$ independent of $(e_0)_x$, and therefore determines a non-zero element of $T_x M$ under $\rho : \mathcal{W}M \to TM$, the anchor map of the Cartan algebroid. By projectivizing we obtain a unique element of $\mathrm{P}T_x M$ corresponding to the simplex element $[\zeta_1]$. Let $\varsigma : \mathcal{C}M \to \mathrm{P}TM$ be the map so defined. We show that ς is the projection map of a principal K_{m+1}-bundle structure on $\mathcal{C}M$. We define a right action of K_{m+1} on $\mathcal{C}M$ as follows. First we consider a transformation of frames (ζ_α) of the form $(\zeta_\alpha) \mapsto (\hat{\zeta}_\alpha)$ where

$$\hat{\zeta}_0 = k_0^0 \zeta_0, \quad \hat{\zeta}_1 = k_1^1 \zeta_1 + k_1^0 \zeta_0, \quad \hat{\zeta}_i = k_i^\alpha \zeta_\alpha$$

where $k_0^0 k_1^1 \det(k_j^i) \neq 0$; we then projectivize. The corresponding transformation of simplices defines an element of K_{m+1}, and $[\hat{\zeta}_\alpha]$ is a Cartan simplex if $[\zeta_\alpha]$ is. We obtain in this way an action of K_{m+1} which is clearly an effective right action. Now $\rho(e_0)$ is the zero section of TM, so the orbit of a point of $\mathcal{C}M$ under the action of K_{m+1} is just a fibre of the projection $\varsigma : \mathcal{C}M \to \mathrm{P}TM$.

We can define local sections of ς as follows. Given coordinates (x^a) on M, and adapted coordinates (x^α) on $\mathcal{V}M$, we obtain the global section e_0 and local sections e_a of $\mathcal{W}M \to M$ as the images of Υ and ∂_a respectively. We can introduce local coordinates on $\mathrm{P}TM$ by taking local coordinates (x^a) on M and by noting that every equivalence class of tangent vectors $u^a \partial/\partial x^a$ for which $u^1 \neq 0$ has a unique representative of the form

$$\frac{\partial}{\partial x^1} + y^i \frac{\partial}{\partial x^i};$$

then (x^a, y^i) are local coordinates on $\mathrm{P}TM$. We are effectively using affine (jet-bundle-like) coordinates on $\mathrm{P}TM$, in which we identify an open subset of the fibre of $\mathrm{P}TM$ with an affine submanifold (a hyperplane) of the corresponding fibre of TM, by $(y^i) \mapsto (1, y^i)$. In terms of such coordinates we set

$$\zeta_0 = e_0, \quad \zeta_1 = e_1 + y^i e_i, \quad \zeta_i = e_i;$$

then $[\zeta_\alpha]$ is a local section of ς. Thus $\varsigma : \mathcal{C}M \to \mathrm{P}TM$ is a principal K_{m+1}-bundle. We call $\mathcal{C}M$ with this bundle structure the projective Cartan bundle.

We showed earlier that the transition function for the Cartan bundle $\mathcal{C}M$ relative to local trivializations (over M) of the form $[e_\alpha]$, $[\hat{e}_\alpha]$ is given by projectivizing the $\mathrm{GL}(m+1)$-valued function G where

$$G = \begin{pmatrix} 1 & -\dfrac{1}{m+1}\dfrac{\partial \log|J|}{\partial x^b} \\ 0 & J_b^a \end{pmatrix}.$$

Now $\zeta_\alpha = Y_\alpha^\beta e_\beta$ where Y_α^β are the components of the locally defined $(m+1) \times (m+1)$-matrix-valued function

$$Y = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & y^2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & y^m & 0 & \cdots & 1 \end{pmatrix}.$$

Then the transition function for $\mathcal{C}M$ relative to local trivializations $[\zeta_\alpha]$, $[\hat{\zeta}_\alpha]$ (over $\mathrm{P}TM$) of the form given above is the projection into $\mathrm{PGL}(m+1)$ of $\hat{Y}^{-1}GY$. It follows from this argument that $\hat{Y}^{-1}GY$ takes its values in K_{m+1}, a fact which is not self-evident, though it can be confirmed by explicit calculation without much difficulty.

We will now give an interpretion of this construction in the light of Cartan's approach.

Cartan's study of projective connections [3] covers both the affine and the general cases, although he describes the latter explicitly only when $m = 2$. In the affine case, he envisages a projective space attached to each point of the manifold. Our interpretation of this is that, for each point $x \in M$, we should study the m-dimensional projective space $\mathrm{P}\mathcal{W}_x M$; this space has a distinguished point $[(e_0)_x]$, the point at which the space is 'attached' to M. The geodesics of the connection are the curves in M whose developments into these projective spaces are straight lines.

In the more general case, we can no longer describe the developments of curves into a single projective space at each point. Instead, we have to use a family of projective spaces at each point, with the family parametrized by the set of rays (1-dimensional subspaces of the tangent space) at that point: the projective spaces will therefore need, not just a distinguished point, but also a distinguished ray through that point. This is consistent with Cartan's view in [3], where he takes as a base manifold not M itself, but instead the 'manifold of elements', where an 'element' is a ray at a point.

This suggests that we should consider the pull-back bundle $\tau_M^{\circ *}(\tau) : \tau_M^{\circ *}(\mathcal{W}M) \to T^\circ M$. The canonical global section $e_0 : M \to \mathcal{W}M$ gives rise to a global section of this pull-back bundle which we will continue to denote by e_0. There is now, however, a distinguished 1-dimensional affine sub-bundle $\mathcal{T}_M \subset \tau_M^{\circ *}(\mathcal{W}M)$, defined by specifying that $(v, \zeta) \in \mathcal{T}_M$ whenever $\rho(\zeta) = v$: here we consider the pull-back as a fibre product $\tau_M^{\circ *}(\mathcal{W}M) = T^\circ M \times_M \mathcal{W}M$. Any section of $\mathcal{T}_M \to T^\circ M$ maps, under ρ, to the total derivative section $\mathbf{T}$ of $\tau_M^{\circ *}(TM) \to T^\circ M$, and any two such sections differ by a multiple of e_0.

We now projectivize this construction, both in the fibre and in the base, to give the pull-back bundle $\pi_M^*(\pi)$: $\pi_M^*(\mathrm{P}\mathcal{W}M) \to \mathrm{P}TM$ where $\pi_M : \mathrm{P}TM \to M$ and $\pi : \mathrm{P}\mathcal{W}M \to M$ are the projective tangent bundle and projective Cartan algebroid respectively. This new bundle also has a global section which we continue to denote by $[e_0]$; thus each projective fibre of $\pi_M^*(\mathrm{P}\mathcal{W}M)$ has a distinguished point, the image of this global section. But now each fibre also has a distinguished line containing that point: we define $\mathrm{P}\mathcal{T}_M \subset \pi_M^*(\mathrm{P}\mathcal{W}M)$ by specifying that $([v], [\zeta]) \in \mathrm{P}\mathcal{T}_M$ if either $[\rho(\zeta)] = [v]$, or else $\rho(\zeta) = 0$ (so that, in the latter case, $[\zeta] = [e_0]_x$, and then $([v], [\zeta])$ is the distinguished point in the fibre at $[v]$). Another way of constructing $\mathrm{P}\mathcal{T}_M$ would be to take the 2-dimensional linear hull of the affine sub-bundle $\mathcal{T}_M$, giving a projective line in each fibre of $\tau_M^{\circ *}(\mathrm{P}\mathcal{W}M) \to T^\circ M$; these lines then map consistently to lines in the fibres of $\pi_M^*(\mathrm{P}\mathcal{W}M) \to \mathrm{P}TM$.

We construct the pull-back bundle $\pi_M^*(\mathcal{S}_\mathcal{W}M) \to \mathrm{P}TM$ in the same way: this is, of course, a principal $\mathrm{PGL}(m+1)$-bundle. Then the projective Cartan bundle $\varsigma : \mathcal{C}M \to \mathrm{P}TM$ is the sub-bundle of $\pi_M^*(\mathcal{S}_\mathcal{W}M) \to \mathrm{P}TM$ containing pairs $([v], [\zeta_\alpha])$ where $[\zeta_0]$ is the distinguished point and $[\zeta_1]$ is some other element of the distinguished line, so that $[\zeta_0] = [e_0]_x$ and $[\rho(\zeta_1)] = [v]$ (here, of course, ζ_0 and ζ_1 must be linearly independent, so the case $\rho(\zeta_1) = 0$ does not arise).

6. The normal Cartan connection for a projective equivalence class of sprays

We show how to construct a normal Cartan connection associated with the projective equivalence class of an arbitrary (not necessarily affine) spray. This will be a Cartan geometry on the projective tangent bundle $\mathrm{P}TM$ of an m-dimensional manifold M, modelled on $\mathrm{P}T(\mathrm{P}^m) = \mathrm{PGL}(m+1)/\mathrm{K}_{m+1}$.

6.1. Construction of the connection

In any coordinate neighbourhood on M, and the corresponding neighbourhood on $T^\circ M$, we have the $\mathfrak{sl}(m+1)$-valued 1-form

$$\tilde{\omega} = \begin{pmatrix} 0 & -\dfrac{1}{m-1}\mathfrak{R}_{bc}\mathrm{d}x^c \\ \mathrm{d}x^a & \Pi^a_{bc}\mathrm{d}x^c \end{pmatrix},$$

defined in terms of quantities determined by the projective equivalence class of sprays. Thus $\tilde{\omega}$ is formally identical to the normal Cartan connection form in the affine case, though of course it is defined on $T^\circ M$, not M. The coefficients Π^a_{bc} and $\mathfrak{R}_{bc}$ are local functions on $T^\circ M$ which are homogeneous of degree 0. We may therefore think of $\tilde{\omega}$ as defined locally on $\mathrm{P}TM$. The transformation rule for such 1-forms under a coordinate transformation on M is exactly the same as that for the affine case, namely

$$\tilde{\omega} = h^{-1}\hat{\tilde{\omega}}h + h^{-1}\mathrm{d}h \quad \text{where } h = \varepsilon_J|J|^{-1/(m+1)}\begin{pmatrix} 1 & -\dfrac{1}{m+1}\dfrac{\partial \log|J|}{\partial x^b} \\ 0 & J^a_b \end{pmatrix}.$$

We now adapt these 1-forms more closely to the underlying manifold $\mathrm{P}TM$ by introducing affine coordinates y^i on the fibres, $i = 2, 3, \ldots, m$, and carrying out a gauge-type transformation with the matrix Y introduced in the last section; that is, we set

$$\omega = Y^{-1}\tilde{\omega}Y + Y^{-1}\mathrm{d}Y \quad \text{where } Y = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & y^2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & y^m & 0 & \cdots & 1 \end{pmatrix}.$$

It follows from the transitivity property of gauge transformations that the effect of a coordinate transformation is to transform $\hat{\omega}$ to $\omega = k^{-1}\hat{\omega}k + k^{-1}\mathrm{d}k$ with $k = \hat{Y}^{-1}hY$. But this is a transition function for the projective Cartan bundle. Thus the collection of local $\mathfrak{sl}(m+1)$-valued 1-forms ω determines a global $\mathfrak{sl}(m+1)$-valued 1-form on the projective Cartan bundle, of which each local 1-form is a representative in an appropriate gauge.

To show that this global $\mathfrak{sl}(m+1)$-valued 1-form is the connection form of a Cartan connection we have to show that each local 1-form ω is non-singular as a linear map $T_p(\mathrm{P}TM) \to \mathfrak{sl}(m+1)/\mathfrak{k}_{m+1}$ (where $\mathfrak{k}_{m+1}$ is the Lie algebra of K_{m+1}, the group of the principal bundle $\mathcal{C}M \to \mathrm{P}TM$). To do so we merely have to give the explicit formula for ω, which is easily found to be

$$\begin{pmatrix} 0 & -\dfrac{1}{m-1}(\mathfrak{R}_{1a} + y^i\mathfrak{R}_{ia})\mathrm{d}x^a & -\dfrac{1}{m-1}\mathfrak{R}_{ia}\mathrm{d}x^a \\ \mathrm{d}x^1 & (\Pi^1_{1a} + y^k\Pi^1_{ka})\mathrm{d}x^a & \Pi^1_{ja}\mathrm{d}x^a \\ \mathrm{d}x^i - y^i\mathrm{d}x^1 & \mathrm{d}y^i + (\Pi^i_{1a} - y^i\Pi^1_{1a} + y^k\Pi^i_{ka} - y^iy^k\Pi^1_{ka})\mathrm{d}x^a & (\Pi^i_{ja} - y^i\Pi^1_{ja})\mathrm{d}x^a \end{pmatrix};$$

the crucial point is that the 1-forms in the positions below the diagonal in the first and second columns constitute a local basis for 1-forms on $\mathrm{P}TM$.

6.2. Some properties of the connection

We have constructed a Cartan connection on the projective Cartan bundle, associated with a projective equivalence class of sprays. By analogy with the affine case we call it the normal Cartan connection of the projective equivalence class. We will next establish some properties of the normal Cartan connection.

One might expect for a Cartan geometry on $\mathrm{P}TM$ modelled on $\mathrm{P}T(\mathrm{P}^m)$ that the projective tangent bundle structures should be compatible. Such an expectation is made explicit in the second part of Cartan's paper [3], for the case $m = 2$. We will now show that the structures are compatible for the geometry we have just constructed.

First, note that any curve in M has a natural lift to $\mathrm{P}TM$ obtained by adjoining, to each point on it, its tangent line at that point. The compatibility conditions are that the development into $\mathrm{P}T(\mathrm{P}^m)$ of a vertical curve in $\mathrm{P}TM$ is vertical, and the development into $\mathrm{P}T(\mathrm{P}^m)$ of a lifted curve in $\mathrm{P}TM$ is a lifted curve. A curve in $\mathrm{P}TM$ is vertical if its tangent vector is annihilated by the $\mathrm{d}x^a$, and a curve in $\mathrm{P}TM$ is a natural lift if its tangent vector is annihilated by the so-called contact forms $\theta^i = \mathrm{d}x^i - y^i\mathrm{d}x^1$. It is easy to see that

$$(\xi^a, \eta^i) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ \xi^1 & 1 & 0 \\ \xi^i & \eta^i & \delta^i_j \end{pmatrix}$$

is a local section of $\mathrm{PGL}(m+1) \to \mathrm{P}T(\mathrm{P}^m)$, and that the corresponding gauged Maurer–Cartan form is

$$\begin{pmatrix} 0 & 0 & 0 \\ \mathrm{d}\xi^1 & 0 & 0 \\ \mathrm{d}\xi^i - \eta^i\mathrm{d}\xi^1 & \mathrm{d}\eta^i & 0 \end{pmatrix}.$$

Let us write the connection form of the normal Cartan connection as

$$\omega = \begin{pmatrix} \omega^0_0 & \omega^0_1 & \omega^0_j \\ \omega^1_0 & \omega^1_1 & \omega^1_j \\ \omega^i_0 & \omega^i_1 & \omega^i_j \end{pmatrix}$$

for convenience. The equations for the development of a curve σ in $\mathrm{P}TM$ into $\mathrm{P}T(\mathrm{P}^m)$ give

$$a\dot{\xi}^1 - b_i(\dot{\xi}^i - \eta^i\dot{\xi}^1) = \langle\dot{\sigma}, \omega^1_0\rangle, \quad c^i_j(\dot{\xi}^j - \eta^j\dot{\xi}^1) = \langle\dot{\sigma}, \omega^i_0\rangle$$

for some functions $a(t)$, $b_i(t)$, $c^i_j(t)$. The compatibility conditions therefore require that if σ is vertical $\langle\dot{\sigma}, \omega^1_0\rangle = \langle\dot{\sigma}, \omega^i_0\rangle = 0$, while if σ is a lift $\langle\dot{\sigma}, \omega^i_0\rangle = 0$. It is clear from the explicit expressions for ω^1_0 and ω^i_0 that these conditions hold.

A geodesic of the connection is a curve whose development satisfies $\dot{\xi}^i - \eta^i\dot{\xi}^1 = 0$ and $\dot{\eta}^i = 0$; that is, a geodesic is a curve whose tangents are annihilated by both $\omega^i_0 = \theta^i$ and ω^i_1. Now

$$\begin{aligned}
\omega^i_1 &= \mathrm{d}y^i + (\Pi^i_{1a} - y^i\Pi^1_{1a} + y^j\Pi^i_{ja} - y^iy^j\Pi^1_{ja})\mathrm{d}x^a \\
&= \mathrm{d}y^i + (\Pi^i_{11} - y^i\Pi^1_{11} + y^j\Pi^i_{j1} - y^iy^j\Pi^1_{j1})\mathrm{d}x^1 \\
&\quad + (\Pi^i_{1j} - y^i\Pi^1_{1j} + y^k\Pi^i_{kj} - y^iy^k\Pi^1_{kj})\mathrm{d}x^j \\
&= \mathrm{d}y^i + (\Pi^i_{11} + 2y^j\Pi^i_{1j} + y^jy^k\Pi^i_{jk} - y^i(\Pi^1_{11} + 2y^j\Pi^1_{1j} + y^jy^k\Pi^1_{jk}))\mathrm{d}x^1 (\mathrm{mod}\ \theta^l).
\end{aligned}$$

But it follows from the formulae obtained earlier giving the Π^a_{bc} in terms of the f^i for a system of second-order differential equations that

$$\Pi^i_{11} + 2y^j\Pi^i_{1j} + y^jy^k\Pi^i_{jk} - y^i(\Pi^1_{11} + 2y^j\Pi^1_{1j} + y^jy^k\Pi^1_{jk}) = -f^i;$$

thus a geodesic is a solution of the system of $m-1$ second-order differential equations

$$\frac{\mathrm{d}^2x^i}{d(x^1)^2} = f^i\left(x^a, \frac{\mathrm{d}x^j}{\mathrm{d}x^1}\right),$$

or in other words that it is a path of the projective class of sprays, parametrized by the coordinate x^1. That is, the geodesics of the Cartan connection are the paths of the projective class of sprays.

We now consider the curvature of the connection. It may most easily be calculated as follows. Recall that $\omega = Y^{-1}\tilde{\omega}Y + Y^{-1}\mathrm{d}Y$ where

$$\tilde{\omega} = \begin{pmatrix} 0 & -\dfrac{1}{m-1}\mathfrak{R}_{bc}\mathrm{d}x^c \\ \mathrm{d}x^a & \Pi^a_{bc}\mathrm{d}x^c \end{pmatrix}.$$

It follows that the curvature Ω of ω is $Y^{-1}\tilde{\Omega}Y$ where $\tilde{\Omega}$ is the curvature of $\tilde{\omega}$, which is an $\mathfrak{sl}(m+1)$-valued 2-form on $\mathrm{P}TM$, which it is easy to show is given (as a 0-homogeneous 2-form on $T^\circ M$) by

$$\tilde{\Omega} = \begin{pmatrix} 0 & -\dfrac{1}{m-1}\left(\mathfrak{R}_{b[c|d]}\mathrm{d}x^c \wedge \mathrm{d}x^d + \mathfrak{R}_{bd,c}\varphi^c \wedge \mathrm{d}x^d\right) \\ 0 & \dfrac{1}{2}P^a_{bcd}\mathrm{d}x^c \wedge \mathrm{d}x^d + D^a_{bcd}\varphi^c \wedge \mathrm{d}x^d \end{pmatrix}.$$

It can be shown, with some effort, that the curvature Ω has the following properties:

- $\Omega^\alpha_0 = 0$;
- Ω^0_1 is semi-basic;
- Ω^i_1 is semi-basic;
- if we set $\Omega^i_j = K^i_{jkl}\mathrm{d}y^k \wedge \theta^l \bmod(\mathrm{d}x^a \wedge \mathrm{d}x^b)$ then $K^k_{kij} = 0$ and $K^k_{ijk} = 0$;
- if we set $\Omega^i_1 = L^i_j\mathrm{d}x^1 \wedge \theta^j \bmod(\theta^k \wedge \theta^l)$ then $L^k_k = 0$;

and that these characterize the normal Cartan connection amongst all possible Cartan connections on $\mathrm{P}TM$ where the projective structures are compatible and the geodesics are the paths of a given projective class of sprays, or the solutions of a corresponding system of second-order differential equations; see [8]. Then K^i_{jkl} and L^i_j are the quantities defined earlier. (There is also a more general approach, using the techniques of parabolic geometry as described in [2] and, more specifically, in [9], which gives different conditions on the curvature Ω but results in the same normal connection form; we shall not consider this alternative here.)

Now the normal connection is not necessarily torsion-free. For it to be so we must have $\Omega^i_1 = 0$. A necessary condition for this to hold is that $L^i_j = 0$. We show that it is also sufficient. We know that if $L^i_j = 0$ then $P^a_{bcd} = 0$. Now Ω^i_1 depends only on the components $\tilde{\Omega}^a_b$, and does so linearly; moreover Ω^i_1 is semi-basic. It follows that if $P^a_{bcd} = 0$ then $\Omega^i_1 = 0$. Thus $\Omega^i_1 = 0$ if and only if $L^i_j = 0$, and the normal connection is torsion-free if and only if the sprays of the corresponding projective equivalence class are isotropic. This is the case discussed by Grossman [13].

6.3. What happens when the spray is projectively affine

If the second-order differential equation field is projectively equivalent to an affine spray, the connection ω should reduce to a connection on M gauge equivalent to the normal projective connection associated with the affine spray. But what could one mean by 'reduce'? We can obtain a connection on M by pulling ω back by any local section of $\mathrm{P}TM \to M$. Of course, different sections will give different reduced connections; the requirement is that the different reduced connections should all be gauge equivalent, with the gauge transformation being taken from the gauge group appropriate to the affine case, namely H_{m+1}; we call such a gauge transformation a gauge transformation of the first kind. This will be the case if, for every transformation ψ of $\mathrm{P}TM$ fibred over the identity, $\psi^*\omega$ is a gauge transform of ω by a gauge transformation of the first kind. An equivalent condition is that for any vector field V vertical with respect to the projection $\mathrm{P}TM \to M$, $\mathcal{L}_V\omega$ should be infinitesimally gauge equivalent to ω by a gauge transformation of the first kind. That is to say, there should be a function H taking its values in $\mathfrak{h}_{m+1}$, the Lie algebra of H_{m+1}, such that $\mathcal{L}_V\omega = [\omega, H] + \mathrm{d}H$ (the equation obtained by differentiating the gauge transformation equation at the identity). Now

$$\mathcal{L}_V\omega = V \lrcorner \mathrm{d}\omega + \mathrm{d}\langle V, \omega\rangle = V \lrcorner \Omega + [\omega, \langle V, \omega\rangle] + \mathrm{d}\langle, \omega\rangle,$$

and $\langle V, \omega\rangle$ takes its values in $\mathfrak{h}_{m+1}$. So if $V \lrcorner \Omega = 0$ then ω satisfies the requisite condition with $H = \langle V, \omega\rangle$.

Now Y, given earlier, does define a gauge transformation of the first kind, and so the argument above applies equally as well to $\tilde{\omega}$ as to ω. Thus the condition for the connection to reduce to M can equivalently be expressed as $V \lrcorner \tilde{\Omega} = 0$. On the face of it, this amounts to two conditions, namely $D^a_{bcd} = 0$ and $\partial\mathfrak{R}_{bd}/\partial u^c = 0$. The first of these, the vanishing of the Douglas tensor, is just the necessary and sufficient condition for the second-order differential equation field to be projectively equivalent to an affine spray. But then Π^a_{bc} is independent of u^d, and therefore $\mathfrak{R}_{bd}$ is independent of u^c.

We have shown that a necessary and sufficient condition for the second-order differential equation field associated with the normal Cartan connection to be projectively equivalent to an affine spray is that the curvature satisfies $V \lrcorner \Omega =$ for all vector fields V on $\mathrm{P}TM$ vertical over M; and that this latter condition is necessary and sufficient for the connection to be reducible to a projective connection of affine type. When the condition $V \lrcorner \Omega = 0$ holds we may choose any local section of $\mathrm{P}TM$ over M to obtain the reduced connection form. The obvious choice is $y^i = 0$, and with this choice the reduced connection is the normal Cartan connection associated with the affine spray in standard form.

7. Projective connections

We now show how the theory of projective connections of Berwald, Thomas and Whitehead fits in with the theory of Cartan, by showing how to construct the normal Cartan connection form of a projective equivalence class of sprays from the *BTW*-connection, at the global level. There is an analogous, but somewhat simpler, construction in the affine case, the details of which can be found in [7].

7.1. The construction in general

We therefore start with the *BTW*-connection on the volume bundle, and we will show how to construct from it the normal Cartan connection as a global Cartan connection form on the projective Cartan bundle $\varsigma : \mathcal{C}M \to \mathrm{P}TM$. The first step of the process can be described in quite general terms.

Consider a manifold $\mathcal{N}$ with reversible spray S and corresponding Berwald connection ∇, such that there is defined on $\mathcal{N}$ a nowhere-vanishing complete vector field Xsuch that

- $\mathcal{N}$ is fibred over an m-dimensional manifold $\mathcal{M}$ where the fibres are the integral curves of X;
- the complete lift X^{C} of X to $T^\circ\mathcal{N}$ satisfies $\mathcal{L}_{X^{\mathrm{C}}} S = 0$;
- the vertical lift X^{V} of X to $T^\circ\mathcal{N}$ satisfies $\mathcal{L}_{X^{\mathrm{V}}} S = X^{\mathrm{C}} - 2\Delta$.

The Lie derivative conditions are modelled on the first two conditions for a *BTW*-spray, of course.

Let $\xi : \mathcal{N} \to \mathcal{M}$ be the projection. Note that the vector fields X^{C} and X^{V} define an integrable distribution on $T\mathcal{N}$ whose leaves are the fibres of the projection $\xi_* : T\mathcal{N} \to T\mathcal{M}$. The inverse image of the zero section of $T\mathcal{M}$ under ξ_*, $\xi_*^{-1}(0)$, is the 1-dimensional vector sub-bundle of $T\mathcal{N}$ spanned by X (considered as a section of $\tau_{\mathcal{N}} : T\mathcal{N} \to \mathcal{N}$). Denote by $T^X\mathcal{N}$ the complement of $\xi_*^{-1}(0)$ in $T\mathcal{N}$; it is an open submanifold of $T\mathcal{N}$, fibred over $\mathcal{N}$, contained in $T^\circ\mathcal{N}$. We denote by $\tau_{\mathcal{N}}^X : T^X\mathcal{N} \to \mathcal{N}$ the restriction of τ_N to $T^X\mathcal{N}$.

We denote by ϕ_t the 1-parameter group on $\mathcal{N}$ whose infinitesimal generator is X.

Let $\mathcal{F}\mathcal{N}$ be the frame bundle of $\mathcal{N}$, $\tau_{\mathcal{N}}^*(\mathcal{F}\mathcal{N})$ its pullback over $T\mathcal{N}$. We define a group structure on $\mathbf{R}^2 \times \mathbf{R}_\circ \times \mathbf{R}_\circ$, where $\mathbf{R}_\circ$ is the multiplicative group of non-zero reals, by $(q, r, s, t) \cdot (q', r', s', t') = (q + q', rs' + r', ss', tt')$. This group acts on $\tau_{\mathcal{N}}^*(\mathcal{F}\mathcal{N})$ to the right by

$$\psi_{(q,r,s,t)} : (x, u, \{e_\alpha\}) \mapsto (\phi_q x, \phi_{q*}(su + rX_x), \{t\phi_{q*}e_\alpha\}).$$

Note that this action is fibred over the action $\bar\psi$ of $\mathbf{R}^2 \times \mathbf{R}_\circ$ on $T\mathcal{N}$ given by

$$\bar\psi_{(q,r,s)} : (x, u) \mapsto (\phi_q x, \phi_{q*}(su + rX_x)).$$

This action leaves $T^X\mathcal{N}$ invariant, and the quotient of $T^X\mathcal{N}$ by it is $\mathrm{P}T\mathcal{M}$. Furthermore, the ψ action commutes with the right action of $\mathrm{GL}(m + 1)$ on $\tau_{\mathcal{N}}^*(\mathcal{F}\mathcal{N})$, and leaves $\tau_{\mathcal{N}}^{X*}(\mathcal{F}\mathcal{N})$ invariant. Let $\mathcal{S}_\psi(\mathrm{P}T\mathcal{M})$ be the quotient of $\tau^{X*}(\mathcal{F}\mathcal{N})$ under the ψ action; it is a principal fibre bundle over $\mathrm{P}T\mathcal{M}$ with group $\mathrm{PGL}(m + 1)$, and for any $a \in \mathrm{GL}(m + 1)$, $\pi^X \circ R_a = R_{o(a)} \circ \pi^X$ where $\pi^X : \tau_{\mathcal{N}}^{X*}(\mathcal{F}\mathcal{N}) \to \mathcal{S}_\psi(\mathrm{P}T\mathcal{M})$ and $o : \mathrm{GL}(m + 1) \to \mathrm{PGL}(m + 1)$ are the projections.

Introduce local coordinates $(x^\alpha, u^\alpha, x^\alpha_\beta)$ on $\tau_{\mathcal{N}}^*(\mathcal{F}\mathcal{N})$, where for a frame $\{e_\alpha\}$, $e_\alpha = x^\beta_\alpha \partial_\beta$. The infinitesimal generator of the 1-parameter group $\psi_{(q,0,1,1)}$ on $\tau_{\mathcal{N}}^*(\mathcal{F}\mathcal{N})$ is the vector field Ψ where

$$\Psi = X^\alpha \frac{\partial}{\partial x^\alpha} + u^\beta \frac{\partial X^\alpha}{\partial x^\beta} \frac{\partial}{\partial u^\alpha} + x^\gamma_\beta \frac{\partial X^\alpha}{\partial x^\gamma} \frac{\partial}{\partial x^\alpha_\beta}.$$

The generator of $\psi_{(0,r,1,1)}$ is

$$\Xi = X^\alpha \frac{\partial}{\partial u^\alpha},$$

while that of $\psi_{(0,0,e^s,1)}$ is

$$\tilde\Delta = u^\alpha \frac{\partial}{\partial u^\alpha};$$

Ξ is formally identical to X^{V}, and $\tilde\Delta$ to Δ, but both are vector fields on $\tau_{\mathcal{N}}^*(\mathcal{F}\mathcal{N})$. The generator of $\psi_{(0,0,1,e^t)}$ is the vertical vector field on the $\mathrm{GL}(m + 1)$-bundle $\tau_{\mathcal{N}}^*(\mathcal{F}\mathcal{N})$ corresponding to the identity matrix $I \in \mathfrak{gl}(m + 1)$, that is

$$I^\dagger = x^\alpha_\beta \frac{\partial}{\partial x^\alpha_\beta}.$$

The pairwise brackets of the vector fields Ψ, Ξ, $\tilde\Delta$ and $I^\dagger$ all vanish except that $[\Xi, \tilde\Delta] = \Xi$. These vector fields, when restricted to $\tau_{\mathcal{N}}^{X*}(\mathcal{F}\mathcal{N})$, are linearly independent, and span an integrable distribution $\mathcal{D}$ there whose leaves are just the orbits of the $\psi_{(q,r,e^s,e^t)}$ action. The distribution is invariant under $\psi_{(0,0,\pm1,\pm1)}$. The leaves of $\mathcal{D}$, quotiented by the action of $\psi_{(0,0,\pm1,\pm1)}$, are the fibres of the projection $\pi^X : \tau_{\mathcal{N}}^{*X}(\mathcal{F}\mathcal{N}) \to \mathcal{S}_\psi(\mathrm{P}T\mathcal{M})$.

With respect to the Berwald connection ∇, any curve σ in $T^\circ\mathcal{N}$ has a horizontal lift σ^{H} to $\tau_N^{\circ*}(\mathcal{F}\mathcal{N})$ starting at a given frame $\{e_\alpha\}$ at $\sigma(0)$, defined as follows: $\sigma^{\mathrm{H}}(t)$ is the frame at $\sigma(t)$ obtained by parallelly transporting $\{e_\alpha\}$ to $\sigma(t)$ along σ; a frame field $\{E_\alpha\}$ along σ is parallel if $\nabla_{\dot\sigma} E_\alpha = 0$. Thus any vector field Z on $T^\circ\mathcal{N}$ has a horizontal lift Z^{H} to $\tau_N^{\circ *}(\mathcal{FN})$; in particular the horizontal lift $(X^{\mathrm{C}})^{\mathrm{H}}$ of X^{C} is given by

$$(X^{\mathrm{C}})^{\mathrm{H}} = X^\alpha \frac{\partial}{\partial x^\alpha} + u^\beta \frac{\partial X^a}{\partial x^\beta}\frac{\partial}{\partial u^\alpha} - x^\gamma_\beta \Gamma^\alpha_{\gamma\delta} X^\delta \frac{\partial}{\partial x^\alpha_\beta},$$

where of course

$$\text{if} \quad S = u^\alpha \frac{\partial}{\partial x^\alpha} - 2\Gamma^\alpha \frac{\partial}{\partial u^\alpha} \quad \text{then} \quad \Gamma^\alpha_{\beta\gamma} = \frac{\partial^2 \Gamma^\alpha}{\partial u^\beta \partial u^\gamma}.$$

Thus

$$\Psi - (X^{\mathrm{C}})^{\mathrm{H}} = x^\gamma_\beta \left(\frac{\partial X^\alpha}{\partial x^\gamma} + \Gamma^\alpha_{\gamma\delta} X^\delta \right) \frac{\partial}{\partial x^\alpha_\beta}.$$

Now for any vector field X on $\mathcal{N}$ and Berwald connection ∇, the condition that $\mathcal{L}_{X^{\mathrm{V}}} S = X^{\mathrm{C}} - 2\Delta$, in coordinates, is

$$\frac{\partial X^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma} X^\gamma = \delta^\alpha_\beta;$$

so this condition is equivalent to

$$\Psi - (X^{\mathrm{C}})^{\mathrm{H}} = I^\dagger.$$

We denote by ω the connection form on $\tau_N^{\circ *}(\mathcal{FN})$ corresponding to ∇; in terms of local coordinates the matrix components of ω are given by

$$\omega^\alpha_\beta = \bar{x}^\alpha_\gamma \Gamma^\gamma_{\delta\epsilon} x^\delta_\beta \mathrm{d}x^\epsilon + \bar{x}^\alpha_\gamma \mathrm{d}x^\gamma_\beta$$

where the matrix $(\bar{x}^\alpha_\beta)$ is the inverse of the matrix (x^α_β). A straightforward calculation shows that the condition $\mathcal{L}_{X^{\mathrm{C}}} S = 0$ entails that $\mathcal{L}_\Psi \omega^\alpha_\beta = 0$ (it would be natural therefore to say that X^{C} is an infinitesimal affine transformation of the Berwald connection). It is also the case that $\mathcal{L}_\Xi \omega^\alpha_b = 0$: we have

$$\mathcal{L}_\Xi \omega^\alpha_\beta = \bar{x}^\alpha_\gamma \left(X^\lambda \frac{\partial \Gamma^\gamma_{\delta\epsilon}}{\partial u^\lambda} \right) x^\delta_\epsilon \mathrm{d}x^\epsilon,$$

and it follows from the coordinate form of the condition $\Psi - (X^{\mathrm{C}})^{\mathrm{H}} = I^\dagger$ above, on differentiating with respect to u^λ, that the coefficient vanishes. Furthermore, $\mathcal{L}_{\tilde{\Delta}} \omega = 0$ by homogeneity, and $\mathcal{L}_{I^\dagger} \omega = [I, \omega] = 0$.

We now restrict to $\tau_{\mathcal{N}}^{X*}(\mathcal{FN})$. We can write any vector field in $\mathcal{D}$ in the form $Z = f\Psi + g\Xi + h\tilde{\Delta} + kI^\dagger$, so that

$$\mathcal{L}_Z \omega = (f\mathcal{L}_\Psi + g\mathcal{L}_\Xi + h\mathcal{L}_{\tilde{\Delta}} + k\mathcal{L}_{I^\dagger})\omega + \langle \Psi, \omega \rangle \mathrm{d}f + \langle \Xi, \omega \rangle \mathrm{d}g + \langle \tilde{\Delta}, \omega \rangle \mathrm{d}h + \langle I^\dagger, \omega \rangle \mathrm{d}k.$$

But $\langle \Xi, \omega \rangle = \langle \tilde{\Delta}, \omega \rangle = 0$, while $\langle \Psi, \omega \rangle = \langle I^\dagger, \omega \rangle = I$. It follows that $\mathcal{L}_Z \omega = I(\mathrm{d}f + \mathrm{d}k)$, that is, for any $Z \in \mathcal{D}$, $\mathcal{L}_Z \omega$ is a multiple of the identity element of $\mathfrak{gl}(m+1)$. Finally, ω is invariant under $\psi_{(0,0,\pm 1,1)}$ because S is reversible by assumption, and under $\psi_{(0,0,1,\pm 1)}$ by inspection.

We can therefore define an $\mathfrak{sl}(m+1)$-valued 1-form $\hat{\omega}$ on $\mathcal{S}_\psi(\mathrm{P}T\mathcal{M})$ as follows: for $Q \in \mathcal{S}_\psi(\mathrm{P}T\mathcal{M})$, $w \in T_Q \mathcal{S}_\psi(\mathrm{P}T\mathcal{M})$,

$$\langle w, \hat{\omega}_Q \rangle = \langle v, o_* \omega_P \rangle$$

for any $P \in \tau_{\mathcal{N}}^{X*}(\mathcal{FN})$ such that $\pi^X(P) = Q$, and any $v \in T_P(\tau_{\mathcal{N}}^{X*}(F\mathcal{N}))$ such that $\pi^X_* v = w$, where $o_* : \mathfrak{gl}(m+1) \to \mathfrak{sl}(m+1)$ is the homomorphism of Lie algebras induced by $o : \mathrm{GL}(m+1) \to \mathrm{PGL}(m+1)$; $\hat{\omega}$ is well-defined because $o_* \omega_P(v)$ is unchanged by a change of choices of P and v satisfying the same conditions. We have $\pi^{X*}\hat{\omega} = o_* \omega$, and so for any $a \in \mathrm{GL}(m+1)$,

$$\begin{aligned} \pi^{X*}(R^*_{o(a)}\hat{\omega}) &= R^*_a(\pi^{X*}\hat{\omega}) = R^*_a(o_*\omega) \\ &= o_*(R^*_a \omega) = o_*(\mathrm{ad}(a^{-1})\omega) = \mathrm{ad}(o(a)^{-1}) o_* \omega \\ &= \pi^{X*}(\mathrm{ad}(o(a)^{-1})\hat{\omega}), \end{aligned}$$

and so since π^X is surjective, $R^*_{o(a)}\hat{\omega} = \mathrm{ad}(o(a)^{-1})\hat{\omega}$. Moreover, for any $A \in \mathfrak{gl}(m+1)$ we have $\pi^X_* A^\dagger = (o_* A)^\dagger$, and therefore

$$\hat{\omega}(o_*(A)^\dagger) = (\pi^X_* \hat{\omega})(A^\dagger) = o_*\omega(A^\dagger) = o_* A.$$

Thus $\hat{\omega}$ is the connection form of an Ehresmann connection on the principal $\mathrm{PGL}(m+1)$-bundle $\mathcal{S}_\psi(\mathrm{P}T\mathcal{M})$.

We now turn to a further consequence of the condition $\mathcal{L}_{X^\mathrm{V}} S = X^\mathrm{C} - 2\Delta$. We can regard X, which is a vector field on $\mathcal{N}$, as a section of $\tau^{\circ *}_{\mathcal{N}} T\mathcal{N}$, and therefore calculate its Berwald covariant differential: using the coordinate form of this condition we find that

$$\nabla X = \mathrm{d}x^\alpha \otimes \frac{\partial}{\partial x^\alpha}.$$

We also have for the total derivative $\mathbf{T} = u^\alpha \partial/\partial x^\alpha$

$$\nabla \mathbf{T} = \varphi^\alpha \otimes \frac{\partial}{\partial x^\alpha}$$

where as we explained in Section 2, φ^α is the 1-form $\mathrm{d}u^\alpha + \Gamma^\alpha_\beta \mathrm{d}x^\beta$, so that $\{\mathrm{d}x^\alpha, \varphi^\alpha\}$ is the local basis of 1-forms on $T^\circ\mathcal{N}$ dual to the local basis $\{H_\alpha, V_\alpha\}$ of vector fields associated with the Berwald connection. With an eye to the description of the structure of $\mathcal{C}M \to \mathrm{P}TM$ given in the previous section, we seek those vector fields η on $T^\circ\mathcal{N}$ with the properties that $\nabla_\eta(fX) = 0$ for some non-vanishing function f, and $\nabla_\eta(g\mathbf{T} + hX) = 0$ for some functions g and h with g non-vanishing; or equivalently, with the properties that $\nabla_\eta X$ is a multiple of X, and $\nabla_\eta \mathbf{T}$ is a linear combination of X and $\mathbf{T}$. Such η must satisfy $\langle \eta, \mathrm{d}x^\alpha \rangle = \lambda X^\alpha$ and $\langle \eta, \varphi^\alpha \rangle = \mu X^\alpha + \nu u^\alpha$. It follows that η must be a linear combination of X^C, X^V and Δ.

Note that X and $\mathbf{T}$ are linearly independent over $T^X\mathcal{N}$. Let us denote by $\mathcal{F}_X\mathcal{N} \subset \tau^{X*}_{\mathcal{N}}(\mathcal{F}\mathcal{N})$ the sub-bundle consisting of those frames whose first member is a multiple of X and whose second member is a linear combination of $\mathbf{T}$ and X. Then at any point $P \in \mathcal{F}_X\mathcal{N}$, we have $H_P \cap T_P(\mathcal{F}_X\mathcal{N}) = \langle (X^\mathrm{C})^\mathrm{H}_P, (X^\mathrm{V})^\mathrm{H}_P, \Delta^\mathrm{H}_P \rangle$, that is, the horizontal subspace at P (the kernel of ω_P) intersects the tangent space to $\mathcal{F}_X\mathcal{N}$ at P in the 3-dimensional subspace spanned by the horizontal lifts of X^C, X^V and Δ to P. But $(X^\mathrm{V})^\mathrm{H} = \Xi$, $\Delta^\mathrm{H} = \tilde{\Delta}$, and $(X^\mathrm{C})^\mathrm{H} = \Psi - I^\dagger$. Thus $\ker \omega_P \cap T_P(\mathcal{F}_X\mathcal{N}) \subset \mathcal{D}_P$; so when we pass to the quotient, at any point $Q \in \pi^X(\mathcal{F}_X\mathcal{N})$ we have $\ker \hat{\omega}_Q \cap T_Q(\pi^X(\mathcal{F}_X\mathcal{N})) = \{0\}$.

We can apply the above results with $\mathcal{N} = \mathcal{V}M$, $X = \Upsilon$, S the *BTW*-spray of a projective equivalence class of sprays. The manifold $\mathcal{M}$ is just M, and $\mathcal{S}_\psi(\mathrm{P}T\mathcal{M})$ is $\pi^*_M(\mathcal{S}_\mathcal{V}M)$, a principal $\mathrm{PGL}(m+1)$-bundle over $\mathrm{P}TM$. Then $\hat{\omega}$ is an Ehresmann connection form on $\pi^*_M(\mathcal{S}_\mathcal{V}M) \to \mathrm{P}TM$. Now the projective Cartan bundle $\mathcal{C}M \to \mathrm{P}TM$ is a sub-bundle of $\pi^*_M(\mathcal{S}_\mathcal{V}M) \to \mathrm{P}TM$. This sub-bundle has codimension $2m - 1 = \dim(\mathrm{P}TM)$, and so the restriction ω of $\hat{\omega}$ to $\mathcal{C}M$ will define a Cartan connection if the intersection (in $T(\pi^*_M(\mathcal{S}_\mathcal{V}M))$) of $\ker \hat{\omega}$ and $T\mathcal{C}M$ contains only zero vectors ([23], Proposition A.3.1; [18]). But $\mathcal{C}M$ is the image in $\pi^*_M(\mathcal{S}_\mathcal{V}M)$ of the sub-bundle of $\tau^{\Upsilon *}_{\mathcal{V}M}\mathcal{F}(\mathcal{V}M)$ consisting of those frames with first element a multiple of Υ and second a linear combination of Υ and $\mathbf{T}$, so this follows from the results of the previous paragraph.

7.2. The connection forms

The Ehresmann connection form $\tilde{\omega}$ of the *BTW*-connection in the coordinate gauge (∂_α) is given by

$$\tilde{\omega}_{(\partial_\alpha)} = \begin{pmatrix} 0 & -\dfrac{1}{m-1} x^0 \mathfrak{R}_{bc} \mathrm{d}x^c \\ (x^0)^{-1}\mathrm{d}x^a & \Pi^a_{bc}\mathrm{d}x^c + \delta^a_b (x^0)^{-1}\mathrm{d}x^0 \end{pmatrix};$$

this is formally the same as in the affine case, but it must be borne in mind that $\tilde{\omega}_{(\partial_\alpha)}$ is a local matrix-valued 1-form on $T^\circ M$ rather than on M; it is semi-basic over $T^\circ M \to M$. We first change the gauge to (Υ, ∂_a); we obtain

$$\tilde{\omega}_{(\Upsilon,\partial_a)} = \begin{pmatrix} (x^0)^{-1}\mathrm{d}x^0 & -\dfrac{1}{m-1} \mathfrak{R}_{bc} \mathrm{d}x^c \\ \mathrm{d}x^a & \Pi^a_{bc}\mathrm{d}x^c + \delta^a_b (x^0)^{-1}\mathrm{d}x^0 \end{pmatrix}$$

$$= ((x^0)^{-1}\mathrm{d}x^0)I + \begin{pmatrix} 0 & -\dfrac{1}{m-1}\mathfrak{R}_{bc}\mathrm{d}x^c \\ \mathrm{d}x^a & \Pi^a_{bc}\mathrm{d}x^c \end{pmatrix}.$$

The Ehresmann connection $\hat{\omega}$ on the simplex bundle $\pi_M^*(\mathcal{S}_\mathcal{W}M)$, in the gauge $[e_\alpha]$, is therefore

$$\hat{\omega}_{[e_\alpha]} = \begin{pmatrix} 0 & -\dfrac{1}{m-1}\mathfrak{R}_{bc}\mathrm{d}x^c \\ \mathrm{d}x^a & \Pi^a_{bc}\mathrm{d}x^c \end{pmatrix}.$$

To obtain the Cartan connection form we need to change the gauge to $[\zeta_\alpha]$ where

$$\zeta_0 = e_0, \quad \zeta_1 = e_1 + y^i e_i, \quad \zeta_i = e_i;$$

the result is

$$\begin{pmatrix} 0 & -\dfrac{1}{m-1}(\mathfrak{R}_{1a} + y^i\mathfrak{R}_{ia})\mathrm{d}x^a & -\dfrac{1}{m-1}\mathfrak{R}_{ia}\mathrm{d}x^a \\ \mathrm{d}x^1 & (\Pi^1_{1a} + y^k\Pi^1_{ka})\mathrm{d}x^a & \Pi^1_{ja}\mathrm{d}x^a \\ \mathrm{d}x^i - y^i\mathrm{d}x^1 & \mathrm{d}y^i + (\Pi^i_{1a} - y^i\Pi^1_{1a} + y^k\Pi^i_{ka} - y^iy^k\Pi^1_{ka})\mathrm{d}x^a & (\Pi^i_{ja} - y^i\Pi^1_{ja})\mathrm{d}x^a \end{pmatrix},$$

as given earlier.

7.3. The general construction in the affine case

Finally, we review the result of Section 6.3 – what happens when the spray is affine – from the present point of view. We have in any case a global Cartan connection form ω on the manifold $\mathcal{C}M$, which satisfies the defining conditions

1. the map $\omega_p : T_p\mathcal{C}M \to \mathfrak{sl}(m+1)$ is an isomorphism for each $p \in \mathcal{C}M$;
2. $R_k^*\omega = \mathrm{ad}(k^{-1})\omega$ for each $k \in \mathrm{K}_{m+1}$; and
3. $\omega(A^\dagger) = A$ for each $A \in \mathfrak{k}_{m+1}$, where $\mathfrak{k}_{m+1}$ is the Lie algebra of K_{m+1} and where $A^\dagger$ is the fundamental vector field corresponding to A.

A Cartan projective connection form in the affine case is a form ϖ on the same manifold, satisfying the same conditions but with H_{m+1} replacing K_{m+1}, and with condition (3) being replaced, explicitly, by

3a. $\varpi(A^\ddagger) = A$ for each $A \in \mathfrak{h}_{m+1}$, where $\mathfrak{h}_{m+1}$ is the Lie algebra of H_{m+1} and where $A^\ddagger$ is the fundamental vector field corresponding to A.

Now ω is the restriction to $\mathcal{C}M$ of the Ehresmann connection form $\hat{\omega}$ on $\pi_M^*(\mathcal{S}_\mathcal{W}M)$; and $\hat{\omega}$, being an $\mathfrak{sl}(m+1)$-valued connection form, satisfies $\hat{\omega}(A^\dagger) = A$ for all $A \in \mathfrak{sl}(m+1)$, and in particular for all $A \in \mathrm{H}_{m+1}$. The submanifold $\mathcal{C}M \subset \pi_M^*(\mathcal{S}_\mathcal{W}M)$ is not invariant under the action of H_{m+1} on $\pi_M^*(\mathcal{S}_\mathcal{W}M)$, and so the restriction of the fundamental vector field $A^\dagger$ to $\mathcal{C}M$ is not tangent to $\mathcal{C}M$ and, in particular, is not the same as $A^\ddagger$. It is, however, easy to check that

$$\hat{\omega}(A^\dagger) = \omega(A^\ddagger)$$

at all points of $\mathcal{C}M$ for any connection form ω arising in this way: thus, in order for ω to be of affine type, it is enough for the condition that $R_h^*\omega = \mathrm{ad}(h^{-1})\omega$ for each $h \in \mathrm{H}_{m+1}$ to be satisfied. The differential version of this condition is that $\mathcal{L}_{A^\ddagger}\omega = [\omega, A]$ for all $A \in \mathrm{H}_{m+1}$; if we express this in terms of the curvature Ω of ω it becomes $A^\ddagger \lrcorner\, \Omega = 0$, so this is the necessary and sufficient condition for ω to be of affine type. Since $\mathfrak{h}_{m+1}/\mathfrak{k}_{m+1}$ parametrizes the vertical subspaces of $\mathrm{P}TM \to M$ at each point, and since necessarily $A^\ddagger \lrcorner\, \Omega = 0$ for $A \in \mathrm{K}_{m+1}$, this condition can be seen to be essentially equivalent to the local one found in Section 6.3.

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